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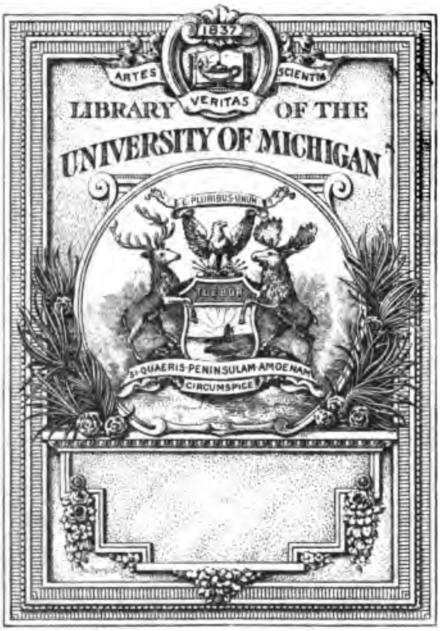
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THE

ELEMENTS

OF

EUCLID;

VIZ.

THE FIRST SIX BOOKS,

TOGETHER WITH THE

ELEVENTH AND TWELFTH.

The ERRORS by which Theon, or others, have long ago
vitiated theſe Books, are Corrected,

And ſome of EUCLID'S Demonstrations are Reſtored.

ALSO

THE BOOK OF

EUCLID'S DATA,

In like Manner Corrected.

By ROBERT SIMSON, M. D.
Emeritus Profeſſor of Mathematics in the Univerſity of Glaſgow.

To this Twelfth Edition are alſo annexed
ELEMENTS OF PLAIN AND SPHERICAL TRIGONOMETRY.

LONDON:
Printed for F. WINGRAVE, in the Strand, Succeſſor to Mr. Nourse.

1804.

Luke Hanſard, Printer, Great Turnſtile, Lincoln's-Inn Fields.

TO THE

KING,

THIS EDITION

OF THE

PRINCIPAL BOOKS

OF THE

ELEMENTS OF EUCLID,

AND OF THE

BOOK OF HIS DATA,

IS MOST HUMBLY DEDICATED,

BY

HIS MAJESTY'S

MOST DUTIFUL,

AND MOST DEVOTED

SUBJECT AND SERVANT,

ROBERT SIMSON.

PREFACE.

THE opinions of the moderns concerning the author of the Elements of Geometry, which go under Euclid's name, are very different and contrary to one another. Peter Ramus aſcribes the Propoſitions, as well as their Demonſtrations, to Theon; others think the Propoſitions to be Euclid's, but that the Demonſtrations are Theon's; and others maintain, that all the Propoſitions and their Demonſtrations are Euclid's own. John Buteo and Sir Henry Savile are the authors of greateſt note who aſſert this laſt; and the greater part of geometers have ever ſince been of this opinion, as they thought it the moſt probable. Sir Henry Savile, after the ſeveral arguments he brings to prove it, makes this concluſion (Page 13. Prælect.) "That, excepting a very few interpolations, explications, and "additions, Theon altered nothing in Euclid." But, by often conſidering and comparing together the Definitions and Demonſtrations as they are in the Greek editions we now have, I found that Theon, or whoever was the editor of the preſent Greek text, by adding ſome things, ſuppreſſing others, and mixing his own with Euclid's Demonſtrations, had changed more things to the worſe than is commonly ſuppoſed, and thoſe not of ſmall moment, eſpecially in the fifth and eleventh Books of the Elements, which this editor has greatly vitiated; for inſtance, by ſubſtituting a ſhorter, but inſufficient Demonſtration of the 18th Prop. of the 5th Book, in place of the legitimate one which Euclid had given; and by taking out of this Book, beſides other things, the good definition which Eudoxus or Euclid had given of compound ratio, and giving an abſurd one in place of it in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Apollonius, nor any geometer before Theon's time, ever made uſe of, and of which there is not to be found the leaſt appearance in any of their writings; and, as this Definition did much embarraſs beginners, and is quite uſeleſs, it is now thrown out of the Elements, and another, which, without doubt, Euclid had given, is put in its proper place among the Definitions of the

5th Book, by which the doctrine of compound ratios is rendered plain and eaſy. Beſides among the Definitions of the 11th Book, there is this, which is the 10th, viz. “Equal and ſimilar ſolid figures are thoſe which are contained by ſimilar planes of the ſame number and magnitude.” Now this Propoſition is a Theorem, not a Definition; becauſe the equality of figures of any kind muſt be demonſtrated, and not aſſumed; and therefore, though this were a true Propoſition, it ought to have been demonſtrated. But, indeed, this Propoſition, which makes the 10th Definition of the 11th Book, is not true univerſally, except in the caſe in which each of the ſolid angles of the figures is contained by no more than three plane angles; for in other caſes, two ſolid figures may be contained by ſimilar planes of the ſame number and magnitude, and yet be unequal to one another, as ſhall be made evident in the Notes ſubjoined to theſe Elements. In like manner, in the Demonſtration of the 26th Prop. of the 11th Book, it is taken for granted, that thoſe ſolid angles are equal to one another which are contained by plain angles of the ſame number and magnitude, placed in the ſame order; but neither is this univerſally true, except in the caſe in which the ſolid angles are contained by no more than three plain angles; nor of this caſe is there any Demonſtration in the Elements we now have, though it be quite neceſſary there ſhould be one. Now, upon the 10th Definition of this Book depend the 25th and 28th Propoſitions of it; and, upon the 25th and 26th depend other eight, viz. the 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame Book; and the 12th of the 12th Book depends upon the eighth of the ſame; and this eighth, and the Corollary of Propoſition 17th and Propoſition 18th of the 12th Book, depend upon the 9th Definition of the 11th Book, which is not a right definition; becauſe there may be ſolids contained by the ſame number of ſimilar plane figures, which are not ſimilar to one another, in the true ſenſe of ſimilarity received by geometers; and all theſe Propoſitions have, for theſe reaſons, been inſufficiently demonſtrated ſince Theon's time hitherto. Beſides, there are ſeveral other things, which have nothing of Euclid's accuracy, and which plainly ſhew, that his Elements have been much corrupted by unſkilful geometers; and, though theſe are not ſo groſs as the others now mentioned, they ought by no means to remain uncorrected.

Upon theſe accounts it appeared neceſſary, and I hope will prove acceptable, to all lovers of accurate reaſoning, and of mathematical learning, to remove ſuch blemiſhes, and reſtore the principal Books of the Elements to their original accuracy, as far as I was able; eſpecially ſince theſe Elements are the foundation of a ſcience by which the inveſtigation and diſcovery of uſeful truths, at leaſt in mathematical learning, is promoted as far as the limited powers of the mind allow; and which likewiſe is of the greateſt uſe in the arts both of peace and war, to many of which geometry is abſolutely neceſſary. This I have endeavoured to do, by taking away the inaccurate and falſe reaſonings which unſkilful editors have put into the place of ſome of the genuine Demonſtrations of Euclid, who has ever been juſtly celebrated as the moſt accurate of geometers, and by reſtoring to him thoſe things which Theon or others have ſuppreſſed, and which have theſe many ages been buried in oblivion.

In this twelfth edition, Ptolemy's Propoſition concerning a property of quadrilateral figures in a circle, is added at the end of the ſixth Book. Alſo the Note on the 29th Propoſition, Book 1ſt, is altered, and made more explicit, and a more general Demonſtration is given, inſtead of that which was in the Note on the 10th Definition of Book 11th; beſides, the Tranſlation is much amended by the friendly aſſiſtance of a learned gentleman.

To which are alſo added, the Elements of Plane and Spherical Trigonometry, which are commonly taught after the Elements of Euclid.

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NOTES, Critical and Geometrical, - - - - - - 287
The DATA, - - - - - - - - - - - - - - 355
Plane and Spherical TRIGONOMETRY, - - - 471

THE ELEMENTS OF EUCLID.

BOOK I.

DEFINITIONS.

I.

A Point is that which hath no parts, or which hath no magnitude. See Notes.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A ſtraight line is that which lies evenly between its extreme points.

V.

A ſuperficies is that which hath only length and breadth.

VI.

The extremities of a ſuperficies are lines.

VII.

A plane ſuperficies is that in which any two points being taken, the ſtraight line between them lies wholly in that ſuperficies. See N.

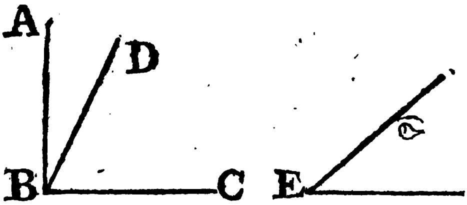
VIII.

" A plane angle is the inclination of two lines to one another in a " plane, which meet together, but are not in the ſame direction." See N.

IX.

A plane rectilineal angle is the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line.

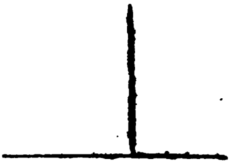
A



‘N.B. When ſeveral angles are at one point B, any one of them is expreſſed by three letters, of which the letter that is at the vertex of the angle, that is at the point in which the ſtraight lines that contain the angle meet one another, is put between the other two letters, and one of theſe two is ſomewhere upon one of thoſe ſtraight lines, and the other upon the other line. thus the angle which is contained by the ſtraight lines AB, CB is named the angle ABC, or CBA; that which is contained by AB, DB is named the angle ABD, or DBA; and that which is contained by DB, CB is called the angle DBC, or CBD. but if there be only one angle at a point, it may be expreſſed by a letter placed at that point; as the angle at E.’

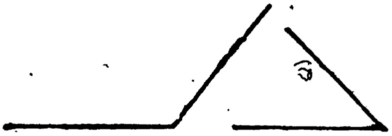
X.

When a ſtraight line ſtanding on another ſtraight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the ſtraight line which ſtands on the other is called a perpendicular to it.



XI.

An obtuſe angle is that which is greater than a right angle.



XII.

An acute angle is that which is leſs than a right angle.

XIII.

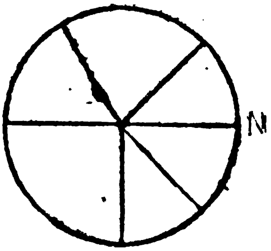
“A term or boundary, is the extremity of any thing.”

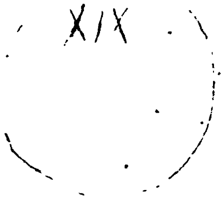
XIV.

A figure is that which is incloſed by one or more boundaries.

XV.

A circle is a plain figure contained by one line, which is called the circumference, and is ſuch that all ſtraight lines drawn from a certain point within the figure to the circumference, are equal to one another.





XVI.

And this point is called the center of the circle.

XVII.

A diameter of a circle is a ſtraight line drawn thro' the center, and terminated both ways by the circumference. See N.

XVIII.

A ſemicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

"A ſegment of a circle is the figure contained by a ſtraight line "and the circumference it cuts off."

XX.

Rectilineal figures are thoſe which are contained by ſtraight lines.

XXI.

Trilateral figures, or triangles, by three ſtraight lines.

XXII.

Quadrilateral, by four ſtraight lines.

XXIII.

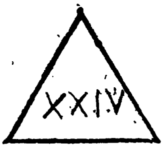
Multilateral figures, or Polygons, by more than four ſtraight lines.

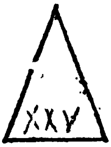
XXIV.

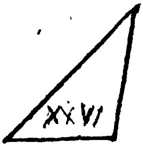
Of three ſided figures, an equilateral triangle is that which has three equal ſides.

XXV.

An iſoſceles triangle, is that which has only two ſides equal.







XXVI.

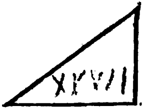
A ſcalene triangle, is that which has three unequal ſides.

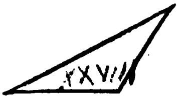
XXVII.

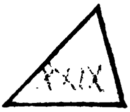
A right angled triangle, is that which has a right angle.

XXVIII.

An obtuſe angled triangle, is that which has an obtuſe angle.





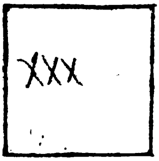


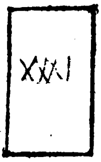
XXIX.

An acute angled triangle, is that which has three acute angles.

XXX.

Of four ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.



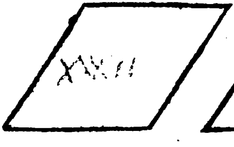


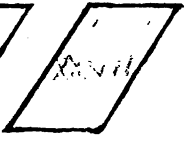
XXXI.

An oblong is that which has all its angles right angles, but has not all its ſides equal.

XXXII.

A rhombus is that which has all its ſides equal, but its angles are not right angles.





XXXIII.

See N. A rhomboid is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXXIV.

All other four ſided figures beſides theſe, are called Trapeziums.

XXXV.

Parallel ſtraight lines are ſuch as are in the ſame plane, and which, being produced ever ſo far both ways, do not meet.

POSTULATES.

I.

LET it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any center, at any diſtance from that center.

AXIOMS.

I.

THINGS which are equal to the ſame are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the ſame, are equal to one another.

VII.

Things which are halves of the ſame, are equal to one another.

VIII.

Magnitudes, which coincide with one another, that is, which exactly fill the ſame ſpace, are equal to one another.

IX.

The whole is greater than its part.

X.

Two ſtraight lines cannot incloſe a ſpace.

XI.

All right angles are equal to one another.

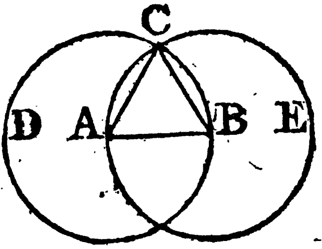
XII.

"If a ſtraight line meets two ſtraight lines, ſo as to make the "two interior angles on the ſame ſide of it taken together leſs "than two right angles, theſe ſtraight lines being continually "produced ſhall at length meet upon that ſide on which are "the angles which are leſs than two right angles. See the "notes on Prop. 29. of Book I."

PROPOSITION I. PROBLEM.

TO deſcribe an equilateral triangle upon a given finite ſtraight line.

Let AB be the given ſtraight line, it is required to deſcribe an equilateral triangle upon it.



From the center A, at the diſtance AB deſcribe [a] the circle BCD. and from the center B, at the diſtance BA deſcribe the circle ACE; and from the point C in which the circles cut one another draw the ſtraight lines [b] CA, CB to the points A, B. ABC ſhall be an equilateral triangle.

a. 3d Poſtulate.

b. 2d Poſt.

Becauſe the point A is the center of the circle BCD, AC is equal [c] to AB. and becauſe the point B is the center of the circle ACE, BC is equal to BA. but it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB. but things which are equal to the ſame are equal to one another [d]; therefore CA is equal to CB. wherefore CA, AB, BC are equal to one another. and the triangle ABC is therefore equilateral, and it is deſcribed upon the given ſtraight line AB. Which was required to be done.

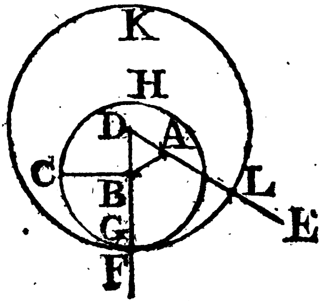
c. 15th Definition.

d. 1ſt Axiom.

PROP. II. PROB.

FROM a given point to draw a ſtraight line equal to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw from the point A a ſtraight line equal to BC.



From the point A to B draw [a] the ſtraight line AB; and upon it deſcribe [b] the equilateral triangle DAB, and produce [c] the ſtraight lines DA, DB to E and F; from the center B, at the diſtance BC deſcribe [d] the circle CGH, and from the center D, at the diſtance DG deſcribe [d] the circle GKL. AL ſhall be equal to BC.

a. 1. Poſt.

b. 1. 1.

c. 2. Poſt.

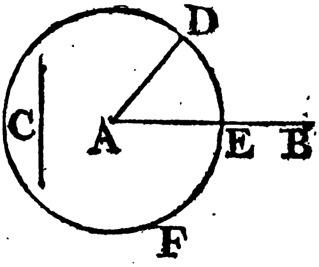
d. 3. Poſt.

Becauſe the point B is the center of the circle CGH, BC is equal[e] to BG. and becauſe D is the center of the circle GKL, DL is equal to DG, and DA, DB parts of them are equal; therefore the remainder AL is equal to the remainder[f] BG. but it has been ſhewn that BC is equal to BG; wherefore AL and BC are each of them equal to BG, and things that are equal to the ſame are equal to one another; therefore the ſtraight line AL is equal to BC. Wherefore from the given point A a ſtraight line AL has been drawn equal to the given ſtraight line BC. Which was to be done.

e. 15. Def.
f. 3. Ax.

PROP. III. PROB.

FROM the greater of two given ſtraight lines to cut off a part equal to the leſs.



Let AB and C be the two given ſtraight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C the leſs.

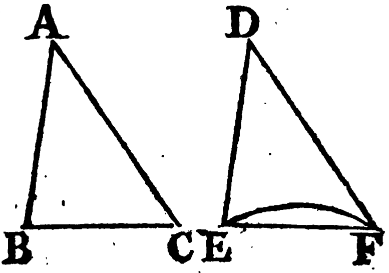
From the point A draw[a] the ſtraight line AD equal to C; and from the center A, and at the diſtance AD deſcribe[b] the circle DEF. and becauſe A is the center of the circle DEF, AE ſhall be equal to AD. but the ſtraight line C is likewiſe equal to AD. whence AE and C are each of them equal to AD. wherefore the ſtraight line AE is equal to[c] C, and from AB the greater of two ſtraight lines, a part AE has been cut off equal to C the leſs. Which was to be done.

a. 2. 1.
b. 3. Poſt.
c. 1. Ax.

PROP. IV. THEOREM.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each; and have likewiſe the angles contained by thoſe ſides equal to one another: they ſhall likewiſe have their baſes, *or third ſides*, equal; and the two triangles ſhall be equal; and their other angles ſhall be equal, each to each, viz. thoſe to which the equal ſides are oppoſite.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF. the baſe BC ſhall be equal to the baſe EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal ſides are oppoſite, ſhall be equal, each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For if the triangle ABC be applied to DEF ſo that the point A may be on D, and the ſtraight line AB upon DE; the point B ſhall coincide with the point E, becauſe AB is equal to DE. and AB coinciding with DE, AC ſhall ~~coincide with~~ DF, becauſe the angle BAC fall on is equal to the angle EDF. wherefore alſo the point C ſhall coincide with the point F, becauſe the ſtraight line AC is equal to DF. but the point B coincides with the point E; wherefore the baſe BC ſhall coincide with the baſe EF. becauſe the point B coinciding with E, and C with F, if the baſe BC does not coincide with the baſe EF, two ſtraight lines would incloſe a ſpace, which is impoſſible [a]. Therefore the baſe BC ſhall coincide with the a. 10. Ax. baſe EF, and be equal to it. Wherefore the whole triangle ABC ſhall coincide with the whole triangle DEF, and be equal to it; and the other angles of the one ſhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore if two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe the angles contained by thoſe ſides equal to one another; their baſes ſhall likewiſe be equal, and the triangles be equal, and their other angles to which the equal ſides are oppoſite, ſhall be equal, each to each. Which was to be demonſtrated.

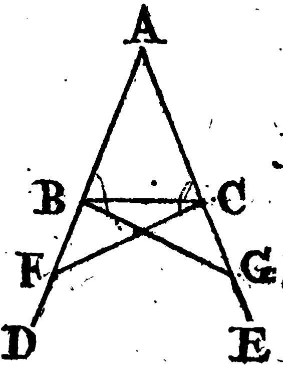
PROP. V. THEOR.

THE angles at the baſe of an Iſoſceles triangle are equal to one another; and if the equal ſides be produced, the angles upon the other ſide of the baſe ſhall be equal.

Let ABC be an Iſoſceles triangle, of which the ſide AB is equal to AC, and let the ſtraight lines AB, AC be produced to D and E. the angle ABC ſhall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE, the greater, cut off AG a. 3. 1. equal[a] to AF, the leſs, and join FC, GB.

Becauſe AF is equal to AG, and AB to AC; the two ſides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the baſe b. 4. 1. FC is equal[b] to the baſe GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal[b] to the remaining angles of the other, each to each, to which the equal ſides are oppoſite; viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB. and becauſe the whole AF is equal to the whole AG, of which the parts AB, AC c. 3. Ax. are equal; the remainder BF ſhall be equal[c] to the remainder CG. and FC was proved to be equal to GB; therefore the two ſides BF, FC are equal to the two CG, GB, each to each; and the angle BFC is equal to the angle CGB; and the baſe BC is common to the two triangles BFC, CGB; wherefore the triangles are equal[b], and their remaining angles, each to each, to which the equal ſides are oppoſite. therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG. and ſince it has been demonſtrated that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF are alſo equal; the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the baſe of the triangle ABC. and it has alſo been proved that the angle FBC is equal to the angle GCB, which are the angles upon the other ſide of the baſe. Therefore the angles at the baſe, &c. Q. E. D.



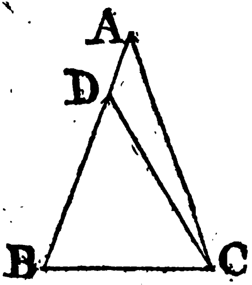
Corollary. Hence every equilateral triangle is alſo equiangular.

PROP. VI. THEOR.

IF two angles of a triangle be equal to one another, the ſides alſo which ſubtend, *or are oppoſite to*, the equal angles ſhall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the ſide AB is alſo equal to the ſide AC.

For if AB be not equal to AC, one of them is greater than the other. let AB be the greater, and from it cut [a] off DB equal to AC, a. 3. 1. the leſs, and join DC. therefore becauſe in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two ſides DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefore the baſe DC is equal to the baſe AB, and the triangle DBC is equal to the triangle [b] ACB, the leſs to the greater; which b. 4. 1. is abſurd. Therefore AB is not unequal to AC, that is, it is equal [illegible] Wherefore if two angles, &c. Q. E. D.



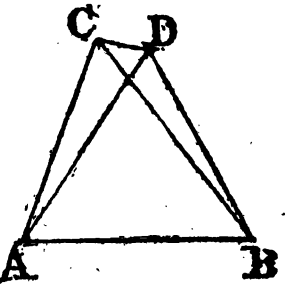
Cor. Hence every equiangular triangle is alſo equilateral.

PROP. VII. THEOR.

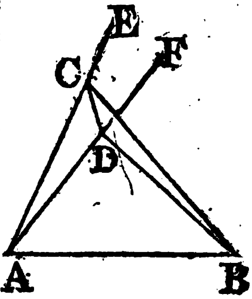
UPON the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. See N.

If it be poſſible, let there be two triangles ACB, ADB upon the ſame baſe AB, and upon the ſame ſide of it, which have their ſides CA, DA, terminated in the extremity A of the baſe, equal to one another, and likewiſe their ſides CB, DB that are terminated in B.

Join CD; then, in the caſe in which the Vertex of each of the triangles is without the other triangle, becauſe AC is equal to AD, the angle ACD is equal [a] to the angle ADC. but the a. 5. 1. angle ACD is greater than the angle BCD, therefore the angle ADC is greater alſo than BCD; much more then is the angle BDC greater than the angle BCD. again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but it has been demonſtrated to be greater than it; which is impoſſible.



But if one of the Vertices, as D, be within the other triangle ACB; produce AC, AD to E, F. therefore becauſe AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the a. 5. 1. other ſide of the baſe CD are equal[a] to one another; but the angle ECD is greater than the angle BCD, wherefore the angle FDC is likewiſe greater than BCD; much more then is the angle BDC greater than the angle BCD. again, becauſe CB is equal to DB, the angle BDC is equal[a] to the angle BCD; but BDC has been proved to be greater than the ſame BCD, which is impoſſible. The caſe in which the Vertex of one triangle is upon a ſide of the other, needs no demonſtration.

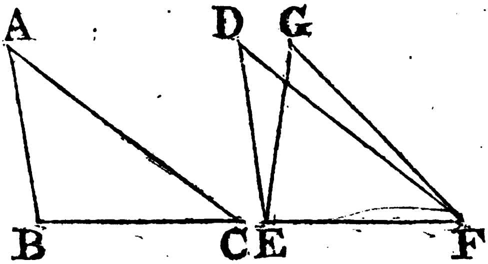


Therefore upon the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe their baſes equal; the angle which is contained by the two ſides of the one ſhall be equal to the angle contained by the two ſides equal to them, of the other.

Let ABC, DEF be two triangles having the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and alſo the baſe BC equal to the baſe EF. The angle BAC is equal to the angle EDF.



For if the triangle ABC be applied to DEF ſo that the point B be on E, and the ſtraight line BC upon EF; the point C ſhall alſo coincide with the point F, becauſe BC is equal to EF.

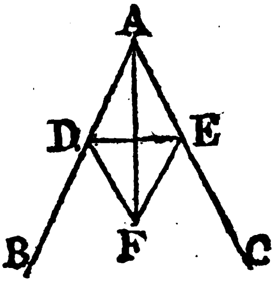
therefore BC coinciding with EF, BA and AC ſhall coincide with ED and DF. for if the baſe BC coincides with the baſe EF, but the ſides BA, CA do not coincide with the ſides ED, FD, but have a different ſituation, as EG, FG; then upon the ſame baſe EF and upon the ſame ſide of it there can be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe their ſides terminated in the other extremity. but this is impoſſible [a]. therefore if the baſe BC coincides with the baſe EF, a. 7. 1.
the ſides BA, AC cannot but coincide with the ſides ED, DF; wherefore likewiſe the angle BAC coincides with the angle EDF, and is equal [b] to it. therefore if two triangles, &c. Q. E. D. b. 8. Ax.

PROP. IX. PROB.

TO biſect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to biſect it.

Take any point D in AB, and from AC cut [a] off AE equal to a. 3. 1.
AD; join DE, and upon it deſcribe [b] b. 1. 1.
an equilateral triangle DEF, then join AF. the ſtraight line AF biſects the angle BAC.



Becauſe AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two ſides DA, AF are equal to the two ſides EA, AF, each to each; and the baſe DF is equal to the baſe EF; therefore the angle DAF is equal
[c] to the angle EAF. wherefore the given rectilineal angle BAC c. 8. 1.
is biſected by the ſtraight line AF. Which was to be done.

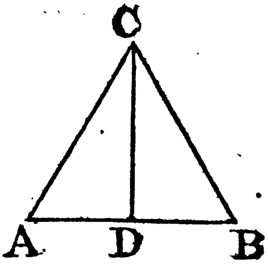
PROP. X. PROB.

TO biſect a given finite ſtraight line, that is, to divide it into two equal parts.

Let AB be the given ſtraight line; it is required to divide it into two equal parts.

Deſcribe [a] upon it an equilateral triangle ABC, and biſect [b] a. 1. 1.
the angle ACB by the ſtraight line CD. AB is cut into two b. 9. 1.
equal parts in the point D.

Becauſe AC is equal to CB, and CD
common to the two triangles ACD,
BCD; the two ſides AC, CD are equal
to BC, CD, each to each; and the angle
ACD is equal to the angle BCD; there-
c. 4. 1. fore the baſe AD is equal to the baſe [c]
DB, and the ſtraight line AB is divided
into two equal parts in the point D.
Which was to be done.

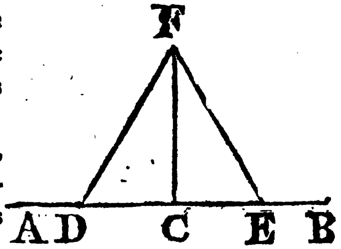


PROP. XI. PROB.

TO draw a ſtraight line at right angles to a given ſtraight line, from a given point in the ſame.

See N. Let AB be a given ſtraight line, and C a point given in it; it is required to draw a ſtraight line from the point C at right angles to AB.

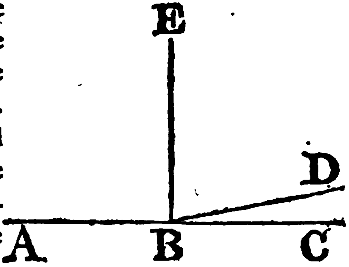
a. 3. 1. Take any point D in AC, and make [a] CE equal to CD, and
b. 1. 1. upon DE deſcribe [b] the equilateral
triangle DFE, and join FC. the
ſtraight line FC, drawn from the
given point C, is at right angles
to the given ſtraight line AB.



Becauſe DC is equal to CE,
and FC common to the two tri-
angles DCF, ECF; the two ſides
DC, CF are equal to the two EC,
CF, each to each; and the baſe DF is equal to the baſe EF;
c. 8. 1. therefore the angle DCF is equal [c] to the angle ECF; and they
are adjacent angles. but when the adjacent angles which one
ſtraight line makes with another ſtraight line are equal to one
d. 10. Def. another each of them is called a right [d] angle; therefore each of
1. the angles DCF, ECF is a right angle. wherefore from the given
point C in the given ſtraight line AB, FC has been drawn at
right angles to AB. Which was to be done.

COR. By help of this Problem it may be demonſtrated that two ſtraight lines cannot have a common ſegment.

If it be poſſible, let the two ſtraight lines ABC, ABD have the ſegment AB common to both of them. from the point B draw BE at right angles to AB; and becauſe ABC is a ſtraight line, the

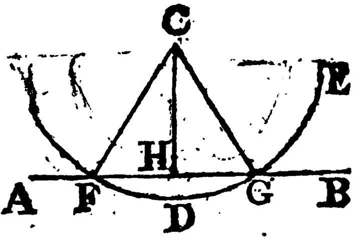


angle CBE is equal [a] to the angle EBA, in the ſame manner, becauſe ABD is a ſtraight line, the angle DBE is equal to the angle EBA. wherefore the angle DBE is equal to the angle CBE, the leſs to the greater; which is impoſſible. therefore two ſtraight lines cannot have a common ſegment. a. 10.Def.1.

PROP. XII. PROB.

TO draw a ſtraight line perpendicular to a given ſtraight line of an unlimited length, from a given point without it.

Let AB be the given ſtraight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a ſtraight line perpendicular to AB from the point C.



Take any point D upon the other ſide of AB, and from the center C, at the diſtance CD, deſcribe [b] the circle EGF meeting AB in F, G; and biſect [c] FG in H, and join CF, CH, CG. the ſtraight line CH drawn from the given point C, is perpendicular to the given ſtraight line AB. b. 3. Poſt c. 10. 1.

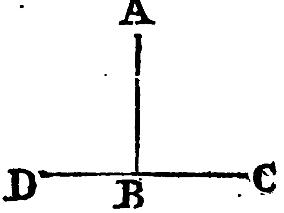
Becauſe FH is equal to HG, and HC common to the two triangles FHC, GHC, the two ſides FH, HC are equal to the two GH, HC, each to each; and the baſe CF is equal [d] to the baſe CG; therefore the angle CHF is equal [e] to the angle CHG; and they are adjacent angles. but when a ſtraight line ſtanding on a ſtraight line makes the adjacent angles equal to one another, each of them is a right angle, and the ſtraight line which ſtands upon the other is called a [f] perpendicular to it. therefore from the given point C a perpendicular CH has been drawn to the given ſtraight line AB. Which was to be done. d. 15.Def.1. e. 8. 1. f. 10.Def.1.

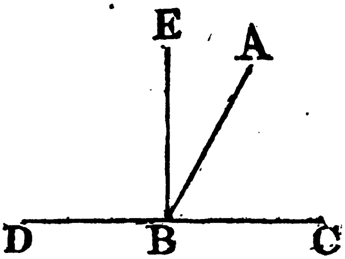
PROP. XIII. THEOR.

THE angles which one ſtraight line makes with another upon one ſide of it, are either two right angles, or are together equal to two right angles.

Let the ſtraight line AB make with CD, upon one ſide of it, the angles CBA, ABD; theſe are either two right angles, or are together equal to two right angles.

a. Def. 10. For if the angle CBA be equal to ABD, each of them is a right [a]



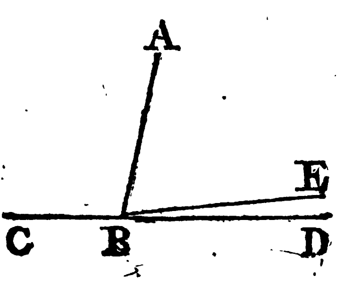


b. 11. 1. angle. but if not, from the point B draw BE at right angles [b] to CD. therefore the angles CBE, EBD are two right angles [a]. and becauſe CBE is equal to the two angles CBA, ABE together; add the angle EBD to each of theſe equals, therefore the angles CBE, EBD are
c. 2. Ax. equal [c] to the three angles CBA, ABE, EBD. again, becauſe the angle DBA is equal to the two angles DBE, EBA, add to theſe equals the angle ABC; therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC. but the angles CBE, EBD have been demonſtrated to be equal to the ſame three angles; and things
d. 1. Ax. that are equal to the ſame are equal [d] to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC. but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore when a ſtraight line, &c. Q. E. D.

PROP. XIV. THEOR.

IF at a point in a ſtraight line, two other ſtraight lines, upon the oppoſite ſides of it, make the adjacent angles together equal to two right angles, theſe two ſtraight lines ſhall be in one and the ſame ſtraight line.

At the point B in the ſtraight line AB, let the two ſtraight lines, BC, BD upon the oppoſite ſides of AB, make the adjacent angles ABC, ABD equal together to two right angles. BD is in the ſame ſtraight line with CB.

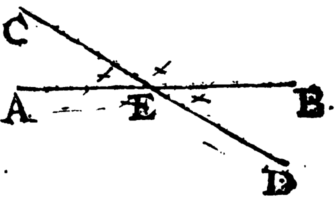


For if BD be not in the ſame ſtraight line with CB, let BE be in the ſame ſtraight line with it. therefore becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one ſide of it, the angles ABC, ABE are together equal [a] to two right angles; a. 13. 1. but the angles ABC, ABD are likewiſe together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD. take away the common angle ABC; the remaining angle ABE is equal [b] to the remaining angle ABD, the leſs to the b. 3. Ax. greater, which is impoſſible. therefore BE is not in the ſame ſtraight line with BC. And in like manner, it may be demonſtrated that no other can be in the ſame ſtraight line with it but BD, which therefore is in the ſame ſtraight line with CB. Wherefore if at a point, &c. Q. E. D.

PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or oppoſite*, angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E. the angle AEC ſhall be equal to the angle DEB, and CEB to AED.



Becauſe the ſtraight line AE makes with CD the angles CEA, AED, theſe angles are together equal [a] to two right angles. again, a. 13. 1. becauſe the ſtraight line DE makes with AB the angles AED, DEB; theſe alſo are together equal [a] to two right angles. and CEA, AED have been demonſtrated to be equal to two right angles; wherefore the angles CEA, AED are equal to the angles AED, DEB. take away the common angle AED, and the remaining angle CEA is equal [b] to the remaining b. 3. Ax. angle DEB. In the ſame manner it can be demonſtrated that the angles CEB, AED are equal. therefore if two ſtraight lines, &c. Q. E. D.

Cor. 1. From this it is manifeſt that if two ſtraight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

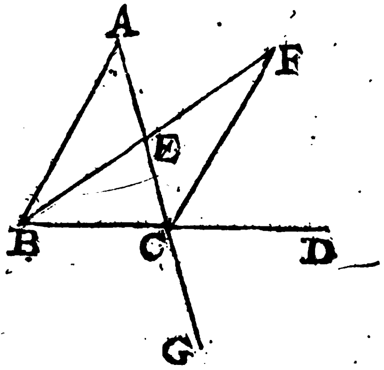
Cor. 2. And conſequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

Book I.

PROP. XVI. THEOR.

IF one ſide of a triangle be produced, the exterior angle is greater than either of the interior oppoſite angles.

Let ABC be a triangle, and let its ſide BC be produced to D. the exterior angle ACD is greater than either of the interior oppoſite angles CBA, BAC.



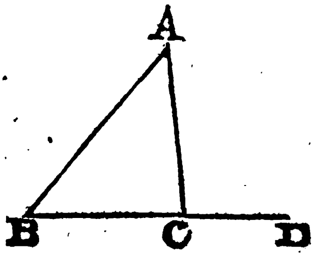
a. 10. 1. Biſect [a] AC in E, join BE and produce it to F, and make EF equal to BE; join alſo FC, and produce AC to G.

b. 15. 1. c. 4. 1. Becauſe AE is equal to EC, and BE to EF; AE, EB are equal to CE, EF, each to each; and the angle AEB is equal [b] to the angle CEF, becauſe they are oppoſite vertical angles. therefore the baſe AB is equal [c] to the baſe CF, and the triangle AEB to the triangle CEF, and the remaining angles, to the remaining angles, each to each, to which the equal ſides are oppoſite. wherefore the angle BAE is equal to the angle ECF. but the angle ECD is greater than the angle ECF, therefore the angle ACD is greater than BAE. in the ſame manner, if the ſide BC be biſected, it may be demonſtrated that the angle

d. 15. 1. BCG, that is [d], the angle ACD, is greater than the angle ABC. therefore if one ſide, &c. Q. E. D.

PROP. XVII. THEOR.

ANY two angles of a triangle are together leſs than two right angles.



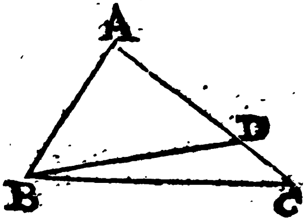
Let ABC be any triangle; any two of its angles together are leſs than two right angles.

a. 16. 1. Produce BC to D; and becauſe ACD is the exterior angle of the triangle ABC, ACD is greater [a] than the interior and oppoſite angle ABC; to each of

theſe add the angle ACB, therefore the angles ACD, ACB are greater than the angles ABC, ACB. but ACD, ACB are together equal [b] to two right angles; therefore the angles ABC, BCA are b. 13. 1.
leſs than two right angles. in like manner it may be demonſtrated that BAC, ACB, as alſo CAB, ABC are leſs than two right angles. therefore any two angles, &c. Q. E. D.

PROP. XVIII. THEOR.

THE greater ſide of every triangle is oppoſite to the greater angle.



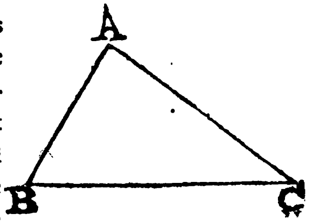
Let ABC be a triangle of which the ſide AC is greater than the ſide AB; the angle ABC is alſo greater than the angle BCA.

Becauſe AC is greater than AB, make [a] AD equal to AB, a. 3. 1.
and join BD. and becauſe ADB is the exterior angle of the triangle BDC, it is greater [b] than the interior and oppoſite angle b. 16. 1.
DCB. but ADB is equal [c] to ABD, becauſe the ſide AB is c. 5. 1.
equal to the ſide AD; therefore the angle ABD is likewiſe greater than the angle ACB; wherefore much more is the angle ABC greater than ACB. therefore the greater ſide, &c. Q. E. D.

PROP. XIX. THEOR.

THE greater angle of every triangle is ſubtended by the greater ſide, or *has the greater ſide oppoſite to it.*

Let ABC be a triangle of which the angle ABC is greater than the angle BCA. the ſide AC is likewiſe greater than the ſide AB.



For if it be not greater, AC muſt either be equal to AB, or leſs than it. it is not equal, becauſe then the angle ABC would be equal [a] to the angle ACB; but it a. 5. 1.
is not; therefore AC is not equal to AB. neither is it leſs; becauſe then the angle ABC would be leſs

[b] than the angle ACB; but it is not; therefore the ſide AC is not leſs than AB. b. 18. 1. and it has been ſhewn that it is not equal to AB. therefore AC is greater than AB. wherefore the greater angle, &c. Q. E. D.

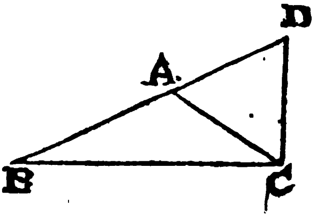
PROP. XX. THEOR.

See N.

ANY two ſides of a triangle are together greater than the third ſide.

Let ABC be a triangle; any two ſides of it together are greater than the third ſide, viz. the ſides BA, AC greater than the ſide BC; and AB, BC greater than AC; and BC, CA greater than AB.

Produce BA to the point D, a. 3. 1. and make [a] AD equal to AC, and join DC.



Becauſe DA is equal to AC, the angle ADC is likewiſe equal b. 5. 1. [b] to ACD. but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC. and becauſe the angle BCD of the tri- c. 19. 1. angle DCB is greater than its angle BDC, and that the greater [c] ſide is oppoſite to the greater angle, therefore the ſide DB is greater than the ſide BC. but DB is equal to BA and AC; therefore the ſides BA, AC are greater than BC. in the ſame manner it may be demonſtrated that the ſides AB, BC are greater than CA; and BC, CA greater than AB. therefore any two ſides, &c. Q. E. D.

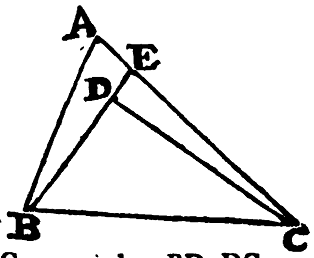
PROP. XXI. THEOR.

See N.

IF from the ends of the ſide of a triangle there be drawn two ſtraight lines to a point within the triangle, theſe ſhall be leſs than the other two ſides of the triangle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C, the ends of the ſide BC of the triangle ABC, to the point D within it. BD and DC are leſs than the other two ſides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and becauſe two ſides of a triangle are greater than the third ſide, the two ſides BA, AE of the triangle ABE are greater than BE. to each of theſe add EC, therefore the ſides BA, AC are greater than BE, EC. again, becauſe the two ſides CE, ED of the triangle CED are greater than CD, add DB to each of theſe; therefore the ſides CE, EB are greater than CD, DB. but it has been ſhewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again becauſe the exterior angle of a triangle is greater than the interior and oppoſite angle, the exterior angle BDC of the triangle CDE is greater than CED. for the ſame reaſon, the exterior angle CEB of the triangle ABE is greater than BAC. and it has been demonſtrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. therefore if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

TO make a triangle of which the ſides ſhall be equal to three given ſtraight lines; but any two whatever of theſe muſt be greater than the third [a].

See N.

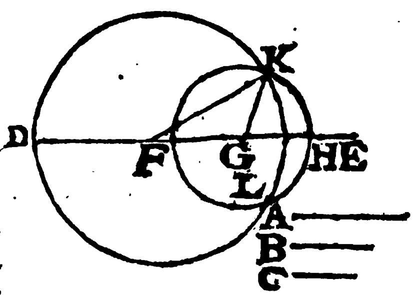
a. 20. 1.

Let A, B, C be the three given ſtraight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the ſides ſhall be equal to A, B, C, each to each.

Take a ſtraight line DE terminated at the point D, but unlimited towards E, and make [a] DF equal to A, FG to B, and GH equal to C; and from the center F, at the diſtance FD deſcribe [b] the circle DKL. and from the center G, at the diſtance GH deſcribe [b] another circle HLK, and join KF, KG. the triangle KFG has its ſides equal to the three ſtraight lines A, B, C.

a. 3. 1.

b. 3. Poſt.



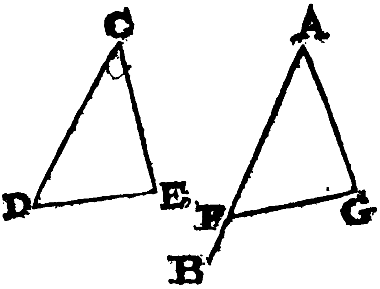
Becauſe the point F is the center of the circle DKL, FD is

Book I. equal [c] to FK; but FD is equal to the ſtraight line A; therefore FK is equal to A. again, becauſe G is the center of the circle LKH, c. 15. Def. GH is equal [c] to GK; but GH is equal to C, therefore alſo GK is equal to C. and FG is equal to B; therefore the three ſtraight lines KF, FG, GK are equal to the three A, B, C. and therefore the triangle KFG has its three ſides KF, FG, GK equal to the three given ſtraight lines A, B, C. Which was to be done.

PROP. XXIII. PROB.

AT a given point in a given ſtraight line to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given ſtraight line AB that ſhall be equal to the given rectilineal angle DCE.



Take in CD, CE, any points D, E, and join DE; a. 22. 1. and make [a] the triangle AFG the ſides of which ſhall be equal to the three ſtraight lines CD, DE, EC, ſo that CD be equal to AF, CE to AG, and DE to FG. and becauſe DC, CE are equal to FA, AG, each to each, and the baſe DE to the baſe FG; the angle b. 8. 1. DCE is equal [b] to the angle FAG. therefore at the given point A in the given ſtraight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

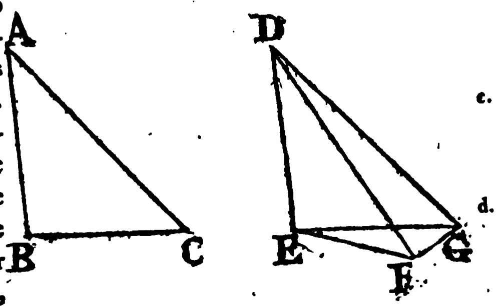
PROP. XXIV. THEOR.

See N. IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the angle contained by the two ſides of one of them greater than the angle contained by the two ſides equal to them, of the other; the baſe of that which has the greater angle ſhall be greater than the baſe of the other.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF. the baſe BC is alſo greater than the baſe EF.

Of the two ſides DE, DF, let DE be the ſide which is not greater than the other, and at the point D in the ſtraight line DE make [a] the angle EDG equal to the angle BAC; and make DG a. 23. 1.
equal [b] to AC or DF, and join EG, GF. b. 3. 1.

Becauſe AB is equal to DE, and AC to DG, the two ſides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the baſe BC is equal [c] to the baſe EG. c. 4. 1.
and becauſe DG is equal to DF, the angle DFG is equal [d] to the angle DGF; d. 5. 1.
but the angle DGF is greater than the angle EGF, therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF. and becauſe the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater [e] ſide is oppoſite to the greater angle; the e. 19. 1.
ſide EG is therefore greater than the ſide EF. but EG is equal to BC; and therefore alſo BC is greater than EF. therefore if two triangles, &c. Q. E. D.



PROP. XXV. THEOR.

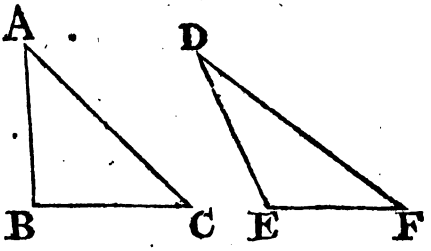
IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the baſe of the one greater than the baſe of the other; the angle alſo contained by the ſides of that which has the greater baſe, ſhall be greater than the angle contained by the ſides equal to them, of the other.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the baſe CB greater than the baſe EF. the angle BAC is likewiſe greater than the angle EDF.

For if it be not greater, it muſt either be equal to it, or leſs. but the angle BAC is not equal to the angle EDF, becauſe then the baſe BC would be equal [a] to EF. but it is not; therefore the angle BAC is not equal to the angle EDF. neither is it leſs; becauſe then the baſe BC would be leſs [b] than the baſe EF; but it is not; therefore the angle BAC is not leſs than the angle EDF, and it was ſhewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore if two triangles, &c. Q. E. D.

a. 4. 1.

b. 24. 1.

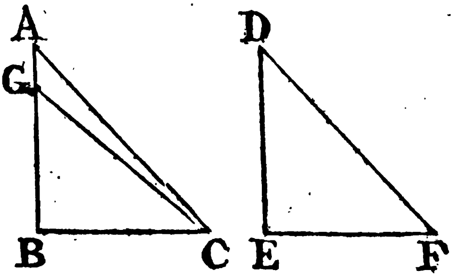


PROP. XXVI. THEOR.

IF two triangles have two angles of one equal to two angles of the other; each to each, and one ſide equal to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to equal angles in each; then ſhall the other ſides be equal, each to each, and alſo the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt, let thoſe ſides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF. the other ſides ſhall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.

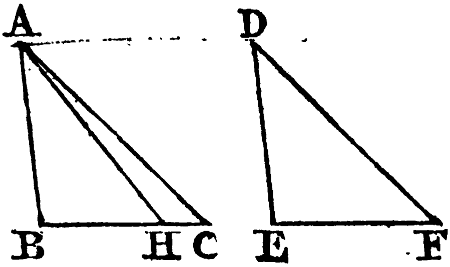
For if AB be not equal to DE, one of them muſt be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC. therefore becauſe BG is



equal to DE, and BC to EF, the two ſides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the baſe GC is equal [a] to the baſe DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal ſides are oppoſite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypotheſis, equal to the angle BCA; wherefore alſo the angle BCG is equal to the angle BCA, the leſs to the greater, which is impoſſible. therefore AB is not unequal to DE, that is, it is equal to it. and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF, the baſe therefore AC is equal [a] to the baſe DF, and the third angle BAC to the third angle EDF.

a. 4. 1.

Next, let the ſides which are oppoſite to equal angles in each triangle be equal to one another, viz. AB to DE; likewiſe in this caſe, the other ſides ſhall be equal, AC to DF, and BC to EF; and alſo the third angle BAC to the third EDF.



For if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH. and becauſe BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the baſe AH is equal to the baſe DF, and the triangle ABH to the triangle DEF, and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite. therefore the angle BHA is equal to the angle EFD. but EFD is equal to the angle BCA; therefore alſo the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and oppoſite angle BCA; which is impoſſible [b]. wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the baſe AC is equal to the baſe DF, and the third angle BAC to the third angle EDF. therefore if two triangles, &c. Q. E. D.

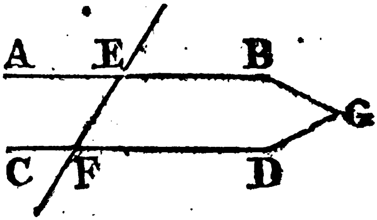
b. 16. 1.

PROP. XXVII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF which falls upon the two ſtraight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For if it be not parallel, AB and CD being produced ſhall meet either towards BD or towards AC. let them be produced and meet towards BD in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater [a] than the interior and oppoſite angle EFG; but it is alſo equal to it, which is impoſſible, therefore AB and CD being produced do not meet towards BD. in like manner it may be demonſtrated that they do not meet towards AC. but thoſe ſtraight lines which meet neither way tho' produced ever ſo far are parallel [b] to one another. AB therefore is parallel to CD. wherefore if a ſtraight line, &c. Q.E.D.



a. 16. 1.

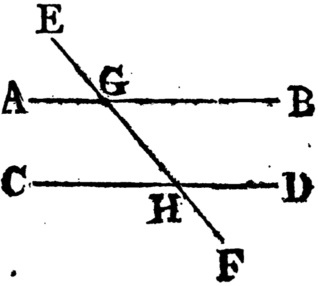
b. 35. Def.

PROP. XXVIII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the exterior angle equal to the interior and oppoſite upon the ſame ſide of the line; or makes the interior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.

Let the ſtraight line EF which falls upon the two ſtraight lines AB, CD make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles. AB is parallel to CD.

Becauſe the angle EGB is equal to the angle GHD, and the angle

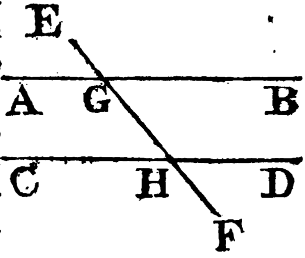


EGB equal [a] to the angle AGH, the angle AGH is equal to the
angle GHD; and they are the alternate angles; therefore AB is
parallel [b] to CD. again, becauſe the angles BGH, GHD are equal [c] a. 15. 1. b. 27. 1.
to two right angles, and that AGH, BGH are alſo equal [d] to two c. By Hyp.
right angles; the angles AGH, BGH are equal to the angles BGH, d. 13. 1.
GHD. take away the common angle BGH, therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. wherefore if a ſtraight line, &c. Q. E. D.

PROP. XXIX. THEOR.

IF a ſtraight line falls upon two parallel ſtraight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and oppoſite upon the ſame ſide; and likewiſe the two interior angles upon the ſame ſide together equal to two right angles. See the Notes on this Propoſition.

Let the ſtraight line EF fall upon the parallel ſtraight lines AB, CD. the alternate angles AGH, GHD are equal to one another; and the exterior angle EGB is equal to the interior and oppoſite, upon the ſame ſide, GHD; and the two interior angles BGH, GHD upon the ſame ſide are together equal to two right angles.



For if AGH be not equal to GHD, one of them muſt be greater than the other; let AGH be the greater. and becauſe the angle AGH is greater than the angle GHD, add to each of them
the angle BGH; therefore the angles AGH, BGH are greater than
the angles BGH, GHD. but the angles AGH, BGH are equal [a] to a. 13. 1.
two right angles; therefore the angles BGH, GHD are leſs than
two right angles. but thoſe ſtraight lines which with another ſtraight
line falling upon them make the interior angles on the ſame ſide
leſs than two right angles, do meet * together if continually pro- * 12. Ax.
duced; therefore the ſtraight lines AB, CD if produced far enough See the
ſhall meet. but they never meet, ſince they are parallel by the Hy- Notes on this Propoſition.
potheſis. therefore the angle AGH is not unequal to the angle GHD,
that is, it is equal to it. but the angle AGH is equal [b] to the angle b. 15. 1.
EGB; therefore likewiſe EGB is equal to GHD. add to each of

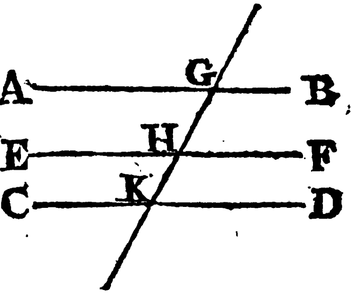
theſe the angle BGH, therefore the angles EGB, BGH are equal to the angles BGH, GHD; but EGB, BGH are equal [c] to two right angles; therefore alſo BGH, GHD are equal to two right angles. wherefore if a ſtraight line, &c. Q. E. D.

c. 13. 1.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the ſame ſtraight line, are parallel to one another.

Let AB, CD be each of them parallel to EF; AB is alſo parallel to CD.



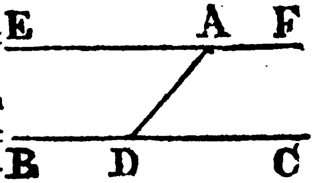
Let the ſtraight line GHK cut AB, EF, CD; and becauſe GHK cuts the parallel ſtraight lines AB, EF, the angle AGH is equal [a] to the angle GHF. again, becauſe the ſtraight line GK cuts the parallel ſtraight lines EF, CD, the angle GHF is equal [a] to the angle GKD. and it was ſhewn that the angle AGK is equal to the angle GHF; therefore alſo AGK is equal to GKD. and they are alternate angles; therefore AB is parallel [b] to CD. wherefore ſtraight lines, &c. Q. E. D.

a. 29. 1.
b. 27. 1.

PROP. XXXI. PROB.

TO draw a ſtraight line thro' a given point parallel to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw a ſtraight line thro' the point A, parallel to the ſtraight line BC.



In BC take any point D, and join AD; and at the point A in the ſtraight line AD make [a] the angle DAE equal to the angle ADC; and produce the ſtraight line EA to F.

a. 23. 1.

Becauſe the ſtraight line AD which meets the two ſtraight lines BC, EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel [b] to BC. therefore the ſtraight line EAF is

b. 27. 1.

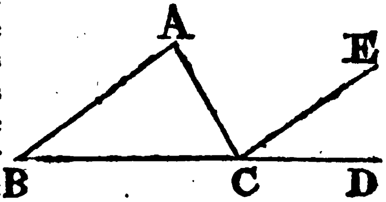
drawn thro' the given point A parallel to the given ſtraight line BC. Which was to be done.

PROP. XXXII. THEOR.

IF a ſide of any triangle be produced, the exterior angle is equal to the two interior and oppoſite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its ſides BC be produced to D. the exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; and the three interior angles of the triangle, viz. ABC, BCA, CAB are together equal to two right angles.

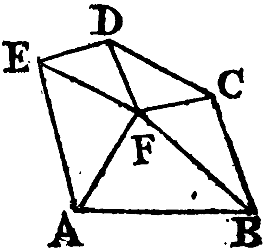
Thro' the point C draw CE parallel [a] to the ſtraight line AB. and becauſe AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal [b]. again becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and oppoſite angle ABC. but the angle ACE was ſhewn to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC. to theſe equals add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB. but the angles ACD, ACB are equal [c] to two right angles; therefore alſo the angles CBA, BAC, ACB are equal to two right angles. wherefore if a ſide of a triangle, &c. Q. E. D.



a. 31. 1.
b. 29. 1.
c. 13. 1.

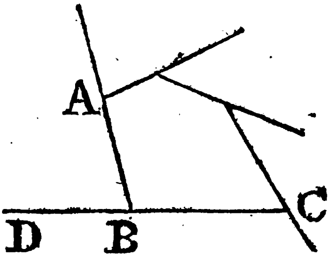
Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure to each of its



angles. And, by the preceding Propoſition, all the angles of theſe triangles are equal to twice as many right angles as there are triangles, that is, as there are ſides of the figure. and the ſame angles are equal to the angles of the figure, together with the angles at a. 2. Cor. 15. 1. the point F which is the common Vertex of the triangles; that is [a], together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

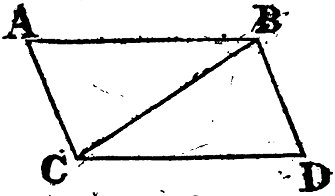
Cor. 2. All the exterior angles of any rectilineal figure are together equal to four right angles.



Becauſe every interior angle ABC with its adjacent exterior b. 13. 1. ABD is equal [b] to two right angles; therefore all the interior together with all the exterior angles of the figure, are equal to twice as many right angles as there are ſides of the figure, that is, by the foregoing Corollary, they are equal to all the interior angles of the figure, together with four right angles. therefore all the exterior angles are equal to four right angles.

PROP. XXXIII. THEOR.

THE ſtraight lines which join the extremities of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.



Let AB, CD be equal and parallel ſtraight lines, and joined towards the ſame parts by the ſtraight lines AC, BD; AC, BD are alſo equal and parallel.

Join BC, and becauſe AB is parallel to CD, and BC meets them; the alternate angles ABC, BCD a. 29. 1. are equal [a]; and becauſe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two ſides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; b. 4. 1. therefore the baſe AC is equal [b] to the baſe BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles[b], each to each, to which the equal ſides are oppoſite. therefore the angle ACB is equal to the angle CBD. and becauſe the ſtraight line BC meets the two ſtraight lines AC, BD and makes the alternate angles ACB, CBD equal to one another, AC is parallel[c] to BD. and it was ſhewn to be equal to it. therefore ſtraight lines, &c. Q. E. D.

b. 4. 1.
c. 27. 1.

PROP. XXXIV. THEOR.

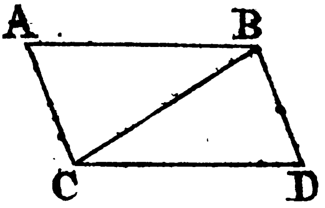
THE oppoſite ſides and angles of parallelograms are equal to one another, and the diameter biſects them, that is, divides them into two equal parts.

N. B. *A Parallelogram is a four ſided figure of which the oppoſite ſides are parallel. and the diameter is the ſtraight line joining two of its oppoſite angles.*

Let ABCD be a parallelogram, of which BC is a diameter. the oppoſite ſides and angles of the figure are equal to one another; and the diameter BC biſects it.

Becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal[a] to one another, and becauſe AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal[a] to one another. wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each, and one ſide BC common to the two triangles, which is adjacent to their equal angles; therefore their other ſides ſhall be equal, each to each, and the third angle of the one to the third angle of the other[b], viz. the ſide AB to the ſide CD, and AC to BD, and the angle BAC equal to the angle BDC. and becauſe the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB; the whole angle ABD is equal to the whole angle ACD. and the angle BAC has been ſhewn to be equal to the angle BDC; therefore the oppoſite ſides and angles of parallelograms are equal to one another. alſo, their diameter biſects them. for, AB being equal to CD, and BC common; the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal to the

a. 29. 1.
b. 26. 1.



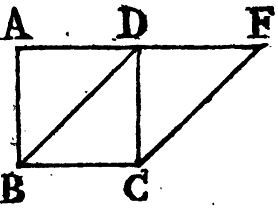
Book I. angle BCD; therefore the triangle ABC is equal [c] to the triangle BCD, and the diameter BC divides the parallelogram ACDB into c. 4. 1. two equal parts. Q. E. D.

PROP. XXXV. THEOR.

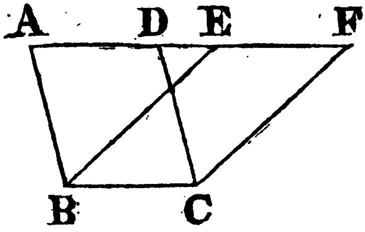
See N. **P**ARALLELOGRAMS upon the ſame baſe and between the ſame parallels, are equal to one another.

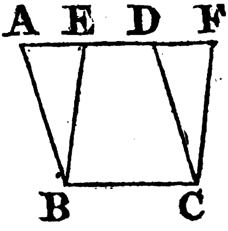
See the 2d and 3d Figures. Let the parallelograms ABCD, EBCF be upon the ſame baſe BC and between the ſame parallels AF, BC. the parallelogram ABCD ſhall be equal to the parallelogram EBCF.

If the ſides AD, DF of the parallelograms ABCD, DBCF oppoſite to the baſe BC, be terminated in the ſame point D; it is plain that each a. 34. 1. of the parallelograms is double [a] of the triangle BDC; and they are therefore equal to one another.



But if the ſides AD, EF oppoſite to the baſe BC of the parallelograms ABCD, EBCF be not terminated in the ſame point; then becauſe ABCD is a parallelogram, AD is equal [a] to BC; for the ſame reaſon, EF is equal to BC; wherefore AD is b. 1. Ax. equal [b] to EF; and DE is common; therefore the whole, or the c. 2. or 3. Ax. remainder, AE is equal [c] to the whole, or the remainder DF; AB alſo is equal to DC; and the two EA, AB are therefore equal



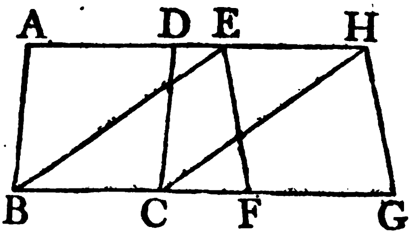


to the two FD, DC, each to each; and the exterior angle FDC d. 29. 1. is equal [d] to the interior EAB; therefore the baſe EB is equal to e. 4. 1. the baſe FC, and the triangle EAB equal [e] to the triangle FDC. take the triangle FDC from the trapezium ABCF, and from the ſame trapezium take the triangle EAB; the remainders therefore are f. 3. Ax. equal [f], that is, the parallelogram ABCD is equal to the parallelogram EBCF. therefore parallelograms upon the ſame baſe, &c. Q. E. D.

PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal baſes and between the ſame parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal baſes BC, FG, and between the ſame parallels AH, BG; the parallelogram ABCD is equal to EFGH.

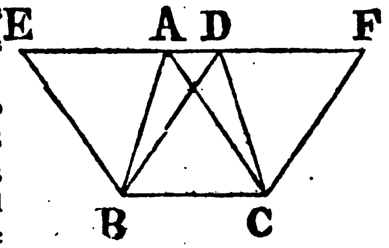


Join BE, CH; and becauſe BC is equal to FG, and FG to [a] EH, BC is a. 34. 1. equal to EH; and they are parallels, and joined towards the ſame parts by the ſtraight lines BE, CH. but ſtraight lines which join equal and parallel ſtraight lines towards the ſame parts, are themſelves equal and parallel [b]; therefore EB, CH are both equal and parallel, and b. 33. 1. EBCH is a parallelogram; and it is equal [c] to ABCD, becauſe it c. 35. 1. is upon the ſame baſe BC, and between the ſame parallels BC, AD. for the like reaſon the parallelogram EFGH is equal to the ſame EBCH. therefore alſo the parallelogram ABCD is equal to EFGH. Wherefore parallelograms, &c. Q. E. D.

PROP. XXXVII. THEOR.

TRIANGLES upon the ſame baſe, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DBC be upon the ſame baſe BC and between the ſame parallels AD, BC. the triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E, F, and thro' B draw [a] BE parallel to CA; a. 31. 1. and thro' C draw [a] CF parallel to BD. therefore each of the figures EBCA, DBCF is a parallelogram; and EBCA is equal [b] to b. 35. 1. DBCF, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, EF; and the triangle ABC is the half of the pa-

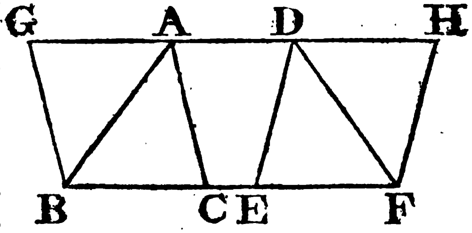
rallelogram EBCA, becauſe the diameter AB biſects [c] it; and the
triangle DBC is the half of the parallelogram DBCF, becauſe
c. 34. 1. the diameter DC biſects it. but the halves of equal things are
d. 7. Ax. equal [d]; therefore the triangle ABC is equal to the triangle DBC.
Wherefore triangles, &c. Q. E. D.

PROP. XXXVIII. THEOR.

TRIANGLES upon equal baſes, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal baſes BC, EF, and between the ſame parallels BF, AD. the triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and thro' B draw
a. 31. 1. BG parallel [a] to CA, and thro' F draw FH parallel to ED. then
each of the figures
GBCA, DEFH is a
parallelogram ; and
b. 36. 1. they are equal [b] to
one another, becauſe
they are upon equal
baſes BC, EF and between the ſame paral-
c. 34. 1. lels BF, GH; and the triangle ABC is the half [c] of the parallelogram GBCA, becauſe the diameter AB biſects it; and the
triangle DEF is the half [c] of the parallelogram DEFH, becauſe
the diameter DF biſects it. but the halves of equal things are
d. 7. Ax. equal [d]; therefore the triangle ABC is equal to the triangle
DEF. Wherefore triangles, &c. Q. E. D.



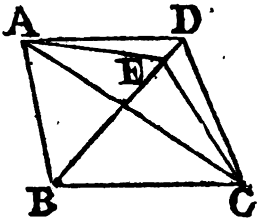
PROP. XXXIX. THEOR.

EQUAL triangles upon the ſame baſe, and upon the ſame ſide of it, are between the ſame parallels.

Let the equal triangles ABC, DBC be upon the ſame baſe BC, and upon the ſame ſide of it; they are between the ſame parallels.

Join AD; AD is parallel to BC; for if it is not, thro' the point
a. 31. 1. A draw [a] AE parallel to BC, and join EC. the triangle ABC is

equal[b] to the triangle EBC, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AE. but the triangle ABC is equal to the triangle BDC; therefore alſo the triangle BDC is equal to the triangle EBC, the greater to the leſs, which is impoſſible. therefore AE is not parallel to BC. in the ſame manner it can be demonſtrated that no other line but AD is parallel to BC; AD is therefore parallel to it. Wherefore equal triangles upon, &c. Q. E. D.



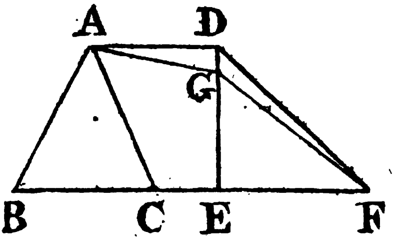
b. 37. 1.

PROP. XL. THEOR.

EQUAL triangles upon equal baſes, and towards the ſame parts, are between the ſame parallels.

Let the equal triangles ABC, DEF be upon equal baſes BC, EF, and towards the ſame parts; they are between the ſame parallels.

Join AD; AD is parallel to BF. for if it is not, thro' A draw[a] AG parallel to BF, and join GF. the triangle ABC is equal[b] to the triangle GEF, becauſe they are upon equal baſes BC, EF, and between the ſame parallels BF, AG. but the triangle ABC is equal to the triangle DEF; therefore alſo the triangle DEF is equal to the triangle GEF, the greater to the leſs, which is impoſſible. therefore AG is not parallel to BF. and in the ſame manner it can be demonſtrated that there is no other parallel to it but AD, AD is therefore parallel to BF. Wherefore equal triangles, &c. Q. E. D.

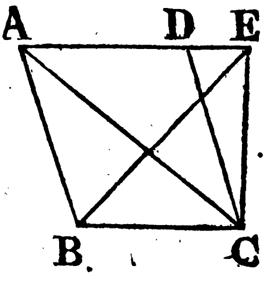


a. 31. 1.
b. 38. 1.

PROP. XLI. THEOR.

IF a parallelogram and triangle be upon the ſame baſe, and between the ſame parallels; the parallelogram ſhall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the ſame baſe BC, and between the ſame parallels BC, AE; the parallelogram ABCD is double of the triangle EBC.

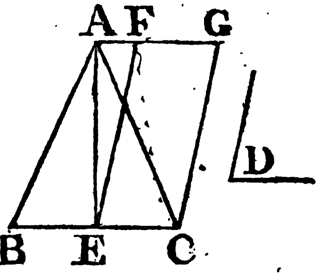


Join AC; then the triangle ABC is
a. 37. 1. equal [a] to the triangle EBC, becauſe
they are upon the ſame baſe BC, and
between the ſame parallels BC, AE.
b. 34. 1. but the parallelogram ABCD is double [b]
of the triangle ABC, becauſe the diameter
AC divides it into two equal
parts; wherefore ABCD is alſo double of the triangle EBC. therefore
if a parallelogram, &c. Q. E. D.

PROP. XLII. PROB.

To deſcribe a parallelogram that ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to deſcribe a parallelogram that ſhall be equal to the given triangle ABC, and have one of its angles equal to D.



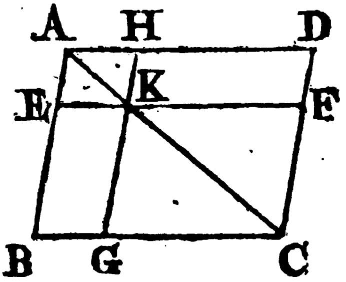
a. 10. 1. Biſect [a] BC in E, join AE, and at the point E in the ſtraight
b. 23. 1. line EC make [b] the angle CEF equal to D; and thro' A draw [c] AG
c. 31. 1. parallel to EC, and thro' C draw [c]
CG parallel to EF. therefore
FECG is a parallelogram. and becauſe
BE is equal to EC, the triangle
d. 38. 1. angle ABE is likewiſe equal [d] to
the triangle AEC, ſince they are
upon equal baſes BE, EC and between
the ſame parallels BC, AG;
therefore the triangle ABC is double
of the triangle AEC. and the parallelogram FECG is likewiſe
e. 41. 1. double [e] of the triangle AEC, becauſe it is upon the ſame baſe,
and between the ſame parallels. therefore the parallelogram FECG
is equal to the triangle ABC, and it has one of its angles CEF
equal to the given angle D. wherefore there has been deſcribed a

parallelogram FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EH, FG the parallelograms about AC, *that is, thro' which* AC *paſſes*, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called the complements. the complement BK is equal to the complement KD.



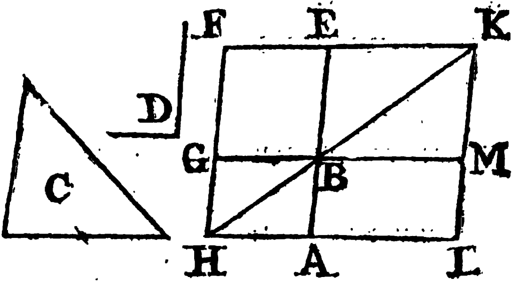
Becauſe ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal [a] to the triangle ADC, and becauſe EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal to the triangle AHK. by the ſame reaſon, the triangle KGC is equal to the triangle KFC. then becauſe the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK together with the triangle KGC is equal to the triangle AHK together with the triangle KFC. but the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD. Wherefore the complements, &c. Q. E. D.

a. 34. 1.

PROP. XLIV. PROB.

TO a given ſtraight line to apply a parallelogram, which ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given ſtraight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the ſtraight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



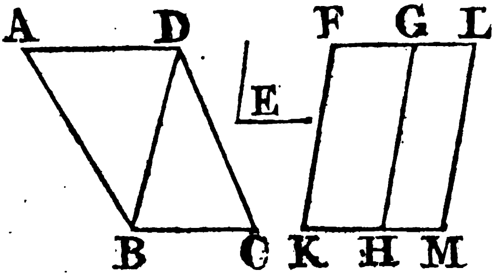
a. 42. 1. Make [a] the parallelogram BEFG equal to the triangle C and having the angle EBG equal to the angle D, ſo that BE be in the ſame ſtraight line with AB, and produce FG to H; and thro' A draw b. 31. 1. [b] AH parallel to BG or EF, and join HB. then becauſe the ſtraight line HF falls upon the parallels AH, EF, the angles AHF, HFE c. 29. 1. are together equal [c] to two right angles; wherefore the angles BHF, HFE are leſs than two right angles. but ſtraight lines which with another ſtraight line make the interior angles upon the d. 12. Ax. ſame ſide leſs than two right angles, do meet [d] if produced far enough. therefore HB, FE ſhall meet, if produced; let them meet in K, and thro' K draw KL parallel to EA or FH, and produce HA, GB to the points L, M. then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK; and LB, BF are the complements; therefore LB is e. 43. 1. equal [e] to BF. but BF is equal to the triangle C; wherefore LB is f. 15. 1. equal to the triangle C. and becauſe the angle GBE is equal [f] to the angle ABM, and likewiſe to the angle D; the angle ABM is equal to the angle D. therefore the parallelogram LB is applied to the ſtraight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D. Which was to be done.

PROP. XLV. PROB.

TO deſcribe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to deſcribe a parallelogram equal to ABCD and having an angle equal to E.

a. 42. 1. Join DB, and deſcribe [a] the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E; b. 44. 1. and to the ſtraight line GH apply [b] the parallelogram GM equal to the triangle DBC having the angle GHM equal to the angle E. and becauſe the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of theſe the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM. but FKH, KHG are equal [c] to two right angles; c. 29. 1. therefore alſo KHG, GHM are equal to two right angles. and becauſe at the point H in the ſtraight line GH, the two ſtraight lines KH, HM upon the oppoſite ſides of it make the adjacent angles equal to two right angles, KH is in the ſame ſtraight [d] line with HM. d. 14. 1. and becauſe the ſtraight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal [c]; add to each of theſe the angle HGL; therefore the angles MHG, HGL are equal to the angles HGF, HGL. but the angles MHG, HGL are equal [c] to two right angles; wherefore alſo the angles HGF, HGL are equal to two right angles, and FG is therefore in the ſame ſtraight line with GL. and becauſe KF is parallel to HG, and HG to ML; KF is parallel [e] to ML. e. 30. 1. and KM, FL are parallels; wherefore KFLM is a parallelogram. and becauſe the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM. therefore the parallelogram KFLM has been deſcribed equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.



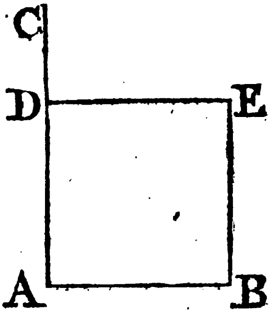
Cor. From this it is manifeſt how to a given ſtraight line to apply a parallelogram, which ſhall have an angle equal to a given rectilineal angle, and ſhall be equal to a given rectilineal figure, viz. by applying [b] to the given ſtraight line, a parallelogram equal b. 44. 1. to the firſt triangle ABD, and having an angle equal to the given angle.

PROP. XLVI. PROB.

TO deſcribe a ſquare upon a given ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe a ſquare upon AB.

a. 11. 1. From the point A draw [a] AC at right angles to AB; and make
b. 3. 1. [b] AD equal to AB, and thro' the point D draw DE parallel [c] to
c. 31. 1. it, and thro' B draw BE parallel to AD. therefore ADEB is a pa-
d. 34. 1. rallelogram; whence AB is equal [d] to DE, and AD to BE. but BA is equal to AD; therefore the four ſtraight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral. likewiſe all its angles are right angles; becauſe the ſtraight line AD meeting the parallels AB, DE, the angles BAD, ADE are
e. 29. 1. equal [e] to two right angles; but BAD is a right angle, therefore alſo ADE is a right angle. but the oppoſite angles of parallelograms are equal [d]; therefore each of the oppoſite angles ABE, BED is a right angle; wherefore the figure ADEB is rectangular. and it has been demonſtrated that it is equilateral; it is therefore a ſquare, and it is deſcribed upon the given ſtraight line AB. Which was to be done.



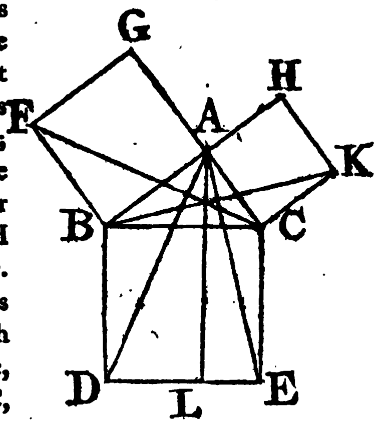
COR. Hence every parallelogram that has one right angle has all its angles right angles.

PROP. XLVII. THEOR.

IN any right angled triangle, the ſquare which is deſcribed upon the ſide ſubtending the right angle, is equal to the ſquares deſcribed upon the ſides which contain the right angle.

Let ABC be a right angled triangle having the right angle BAC; the ſquare deſcribed upon the ſide BC, is equal to the ſquares deſcribed upon BA, AC.

a. 46. 1. On BC deſcribe [a] the ſquare BDEC, and on BA, AC the ſquares



GB, HC; and thro' A draw [b] AL parallel to BD or CE, and join AD, FC. then becauſe each of the angles BAC, BAG is a right angle [c], the two ſtraight lines AC, AG upon the oppoſite ſides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in the ſame ſtraight line [d] with AG. for the ſame reaſon, AB and AH are in the ſame ſtraight line. and becauſe the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle DBA is equal [e] to the whole FBC. and becauſe the two ſides AB, BD are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC; therefore the baſe AD is equal [f] to the baſe FC, and the triangle ABD to the triangle FBC. now the parallelogram BL is double [g] of the triangle ABD, becauſe they are upon the ſame baſe BD, and between the ſame parallels BD, AL; and the ſquare GB is double of the triangle FBC, becauſe theſe alſo are upon the ſame baſe FB, and between the ſame parallels FB, GC. but the doubles of equals are equal [h] to one another. therefore the parallelogram BL is equal to the ſquare GB. and in the ſame manner, by joining AE, BK, it is demonſtrated that the parallelogram CL is equal to the ſquare HC. Therefore the whole ſquare BDEC is equal to the two ſquares GB, HC. and the ſquare BDEC is deſcribed upon the ſtraight line BC, and the ſquares GB, HC upon BA, AC. wherefore the ſquare upon the ſide BC is equal to the ſquares upon the ſides BA, AC. Therefore in any right angled triangle, &c. Q. E. D.

b. 31. 1.
c. 30. Def.
d. 14. 1.
e. 2. Ax.
f. 4. 1.
g. 41. 1.
h. 6. Ax.

PROP. XLVIII. THEOR.

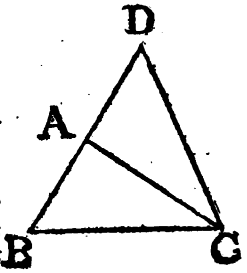
IF the ſquare deſcribed upon one of the ſides of a triangle, be equal to the ſquares deſcribed upon the other two ſides of it; the angle contained by theſe two ſides is a right angle.

If the ſquare deſcribed upon BC one of the ſides of the triangle ABC be equal to the ſquares upon the other ſides BA, AC; the angle BAC is a right angle.

a. 11. 1. From the point A draw [a] AD at right angles to AC, and make AD equal to BA, and join DC. then becauſe DA is equal to AB, the ſquare of DA is equal to the ſquare of AB; to each of theſe add the ſquare of AC, therefore the ſquares of DA, AC are equal to the ſquares of BA, AC. but the ſquare of DC is equal [b] to the ſquares of DA, AC, becauſe DAC is a right angle; and the ſquare of BC, by Hypotheſis, is equal to the ſquares of BA, AC; therefore the ſquare of DC is equal to the ſquare of BC; and therefore alſo the ſide DC is equal to the ſide BC. and becauſe the ſide DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC; and the baſe DC is equal to the baſe BC; therefore the angle DAC is equal [c] to the angle BAC. but DAC is a right angle, therefore alſo BAC is a right angle. Therefore if the ſquare, &c. Q. E. D.

b. 47. 1.

c. 8. 1.



THE

ELEMENTS

OF

EUCLID.

BOOK II.

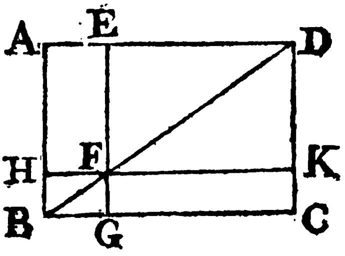
DEFINITIONS.

I.

EVERY right angled parallelogram is ſaid to be contained by any two of the ſtraight lines which contain one of the right angles.

II.

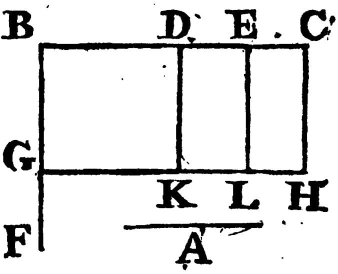
In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. 'Thus the parallelogram HG together with the complements AF, FC is the gnomon, which is more briefly expreſſed by the letters AGK, or EHC which are at the oppoſite angles of the parallelograms which make the gnomon.'



PROP. I. THEOR.

IF there be two ſtraight lines, one of which is divided into any number of parts; the rectangle contained by the two ſtraight lines, is equal to the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Let A and BC be two ſtraight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the ſtraight lines A, BC is equal to the rectangle contained by A, BD; and to that contained by A, DE; and alſo to that contained by A, EC.

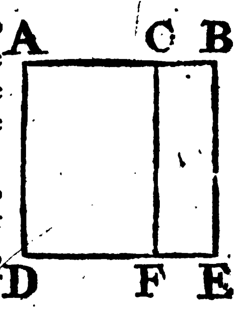


a. 11. 1. From the point B draw [a] BF
at right angles to BC, and make
b. 3. 1. BG equal [b] to A; and thro' G
c. 31. 1. draw [c] GH parallel to BC; and
thro' D, E, C draw [c] DK, EL, CH parallel to BG. then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained
d. 34. 1. by A, DE, becauſe DK, that is [d] BG, is equal to A; and in like manner the rectangle EH is contained by A, EC. therefore the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE, and alſo by A, EC. Wherefore if there be two ſtraight lines, &c. Q. E. D.

PROP. II. THEOR.

If a ſtraight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the ſquare of the whole line.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC together with the rectangle [*] AB, AC ſhall be equal to the ſquare of AB.



a. 46. 1. Upon AB deſcribe [a] the ſquare ADEB,
b. 31. 1. and thro' C draw [b] CF parallel to AD or
BE. then AE is equal to the rectangles AF, CE; and AE is the ſquare of AB; and AF

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[*] N. B. To avoid repeating the word *Contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

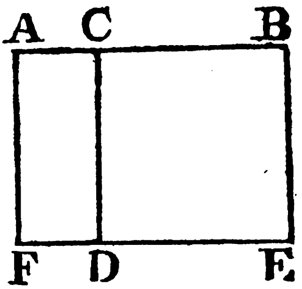
is the rectangle contained by BA, AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB. therefore the rectangle contained by AB, AC together with the rectangle AB, BC, is equal to the ſquare of AB. If therefore a ſtraight line, &c. Q. E. D.

PROP. III. THEOR.

IF a ſtraight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the ſquare of the foreſaid part.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB together with the ſquare of BC.

Upon BC deſcribe [a] the ſquare CDEB, and produce ED to F, and thro' A draw [b] AF parallel to CD or BE. then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to CB; and DB is the ſquare of BC. therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the ſquare of BC. If therefore a ſtraight line, &c. Q. E. D.



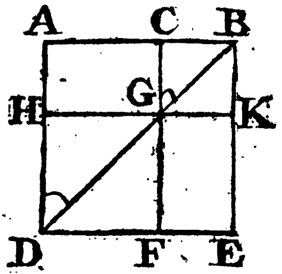
a. 46. 1.
b. 31. 1.

PROP. IV. THEOR.

IF a ſtraight line be divided into any two parts, the ſquare of the whole line is equal to the ſquares of the two parts, together with twice the rectangle contained by the parts.

Let the ſtraight line AB be divided into any two parts in C; the ſquare of AB is equal to the ſquares of AC, CB and to twice the rectangle contained by AC, CB.

Upon AB deſcribe [a] the ſquare ADEB, and join BD, and
thro' C draw [b] CGF parallel to AD or BE, and thro' G draw HK
a. 46. 1. parallel to AB or DE. and becauſe CF is parallel to AD; and BD
b. 31. 1. falls upon them, the exterior angle BGC is equal [c] to the interior
c. 29. 1. and oppoſite angle ADB; but ADB is equal [d] to the angle ABD,
d. 5. 1. becauſe BA is equal to AD, being ſides of a ſquare; wherefore the
angle CGB is equal to the angle GBC,
e. 6. 1. and therefore the ſide BC is equal [e] to
f. 34. 1. the ſide CG. but CB is equal alſo [f] to
GK, and CG to BK; wherefore the
figure CGKB is equilateral. it is likewiſe
rectangular; for CG is parallel to
BK, and CB meets them, the angles
KBC, GCB are therefore equal to two
right angles; and KBC is a right angle,
wherefore GCB is a right angle; and therefore alſo the angles [f]
CGK, GKB oppoſite to theſe are right angles, and CGKB
is rectangular. but it is alſo equilateral, as was demonſtrated;
wherefore it is a ſquare, and it is upon the ſide CB. for the ſame
reaſon HF alſo is a ſquare, and it is upon the ſide HG which is
equal to AC. therefore HF, CK are the ſquares of AC, CB. and
g. 43. 1. becauſe the complement AG is equal [g] to the complement GE,
and that AG is the rectangle contained by AC, CB, for GC is
equal to CB; therefore GE is alſo equal to the rectangle AC,
CB; wherefore AG, GE are equal to twice the rectangle AC,
CB. and HF, CK are the ſquares of AC, CB; wherefore the four
figures HF, CK, AG, GE are equal to the ſquares of AC, CB
and to twice the rectangle AC, CB. but HF, CK, AG, GE make
up the whole figure ADEB which is the ſquare of AB. therefore
the ſquare of AB is equal to the ſquares of AC, CB and twice
the rectangle AC, CB. Wherefore if a ſtraight line, &c.
Q. E. D.



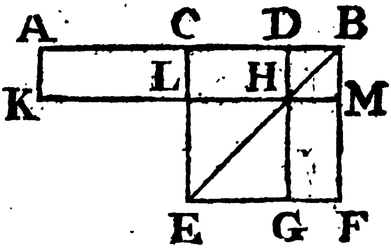
Cor. From the demonſtration it is manifeſt, that the parallelograms about the diameter of a ſquare are likewiſe ſquares.

PROP. V. THEOR.

IF a ſtraight line be divided into two equal parts, and alſo into two unequal parts; the rectangle contained by the unequal parts, together with the ſquare of the line between the points of ſection, is equal to the ſquare of half the line.

Let the ſtraight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB together with the ſquare of CD, is equal to the ſquare of CB.

Upon CB deſcribe [a] the ſquare CEFB, join BE, and thro' D draw [b] DHG parallel to CE or BF; and thro' H draw KLM parallel to CB or EF; and alſo thro' A draw AK parallel to CL or BM. and becauſe the complement CH is equal [c] to the complement HF, to each of theſe add DM, therefore the whole CM is equal to the whole DF; but CM is equal [d] to AL, becauſe AC is equal to CB; therefore alſo AL is equal to DF. to each of theſe add CH, and the whole AH is equal to DF and CH. but AH is the rectangle contained by AD, DB, for DH is equal [e] to DB; and DF together with CH is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD, DB. to each of theſe add LG, which is equal [e] to the ſquare of CD, therefore the gnomon CMG together with LG is equal to the rectangle AD, DB together with the ſquare of CD. but the gnomon CMG and LG make up the whole figure CEFB, which is the ſquare of CB. therefore the rectangle AD, DB together with the ſquare of CD is equal to the ſquare of CB. Wherefore if a ſtraight line, &c. Q. E. D.



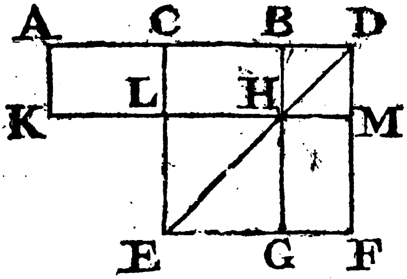
a. 46. 1.
b. 31. 1.
c. 43. 1.
d. 36. 1.
e. Cor. 4. 2.

PROP. VI. THEOR.

IF a ſtraight line be biſected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the ſquare of half of the line biſected, is equal to the ſquare of the ſtraight line which is made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the rectangle AD, DB together with the ſquare of CB, is equal to the ſquare of CD.

a. 46. 1. Upon CD deſcribe [a] the ſquare CEFD, join DE, and thro' B
b. 31. 1. draw [b] BHG parallel to CE or DF, and thro' H draw KLM parallel to AD or EF, and alſo thro' A draw AK parallel to CL or DM.
c. 36. 1. and becauſe AC is equal to CB, the rectangle AL is equal [c] to
d. 43. 1. CH; but CH is equal [d] to HF; therefore alſo AL is equal to HF. to each of theſe add CM, therefore the whole AM is equal to the gnomon CMG. and AM is the rectangle contained by AD, DB, for
e. Cor. 4. 2. DM is equal [e] to DB.
therefore the gnomon CMG is equal to the rectangle AD, DB. add to each of theſe LG, which is equal to the ſquare of CB; therefore the rectangle AD, DB together with the ſquare of CB is equal to the gnomon CMG and the figure LG. but the gnomon CMG and LG make up the whole figure CEFD, which is the ſquare of CD; therefore the rectangle AD, DB together with the ſquare of CB, is equal to the ſquare of CD. Wherefore if a ſtraight line, &c. Q. E. D.

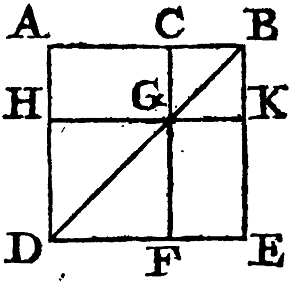


PROP. VII. THEOR.

IF a ſtraight line be divided into any two parts, the ſquares of the whole line, and of one of the parts are equal to twice the rectangle contained by the whole and that part, together with the ſquare of the other part.

Let the ſtraight line AB be divided into any two parts in the point C; the ſquares of AB, BC are equal to twice the rectangle AB, BC together with the ſquare of AC.

Upon AB deſcribe[a] the ſquare ADEB, and conſtruct the figure as in the preceding Propoſitions. and becauſe AG is equal[b] to GE, add to each of them CK, the whole AK is therefore equal to the whole CE; therefore AK, CE are double of AK. but AK, CE are the gnomon AKF together with the ſquare CK; therefore the gnomon AKF together with the ſquare CK is double of AK. but twice the rectangle AB, BC is double of AK, for BK is equal[c] to BC. therefore the gnomon AKF together with the ſquare CK is equal to twice the rectangle AB, BC. to each of theſe equals add HF, which is equal to the ſquare of AC; therefore the gnomon AKF together with the ſquares CK, HF is equal to twice the rectangle AB, BC and the ſquare of AC. but the gnomon AKF together with the ſquares CK, HF make up the whole figure ADEB and CK, which are the ſquares of AB and BC. therefore the ſquares of AB and BC are equal to twice the rectangle AB, BC together with the ſquare of AC. Wherefore if a ſtraight line, &c. Q. E. D.



[a] 46. 1.

[b] 43. 1.

[c] Cor. 4. 2.

PROP. VIII. THEOR.

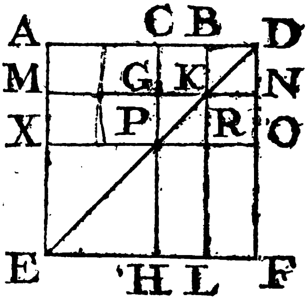
IF a ſtraight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the ſquare of the other part, is equal to the ſquare of the ſtraight line which is made up of the whole and that part.

Let the ſtraight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of the ſtraight line made up of AB and BC together.

Produce AB to D ſo that BD be equal to CB, and upon AD deſcribe the ſquare AEFD; and conſtruct two figures ſuch as in the preceding. Becauſe CB is equal to BD, and that CB is equal[a]

[a] 34. 1.

to GK, and BD to KN; therefore GK is equal to KN. for the ſame reaſon PR is equal to RO. and becauſe CB is equal to BD, and GK to KN, the rectangle CK is equal [b] to BN, and GR to RN. but CK is equal [c] to RN, becauſe they are the complements of the parallelogram CO; therefore alſo BN is equal to GR. and the four rectangles BN, CK, GR, RN, are therefore equal to one another, and ſo are quadruple of one of them CK. again, becauſe CB is equal to BD, and that BD is equal [d] to BK, that is to CG; and CB equal to GK, that [d] is to GP; therefore CG is equal to GP. and becauſe CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF. but MP is equal [c] to PL, becauſe they are the complements of the parallelogram ML; wherefore AG alſo is equal to RF. therefore the four rectangles AG, MP, PL, RF are equal to one another, and ſo are quadruple of one of them AG. And it was demonſtrated that the four CK, BN, GR, RN are quadruple of CK. therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK. and becauſe AK is the rectangle contained by AB, BC, for BK is equal to BC; four times the rectangle AB, BC is quadruple of AK. but the gnomon AOH was demonſtrated to be quadruple of AK; therefore four times the rectangle AB, BC is equal to the gnomon AOH. to each of theſe add XH, which is equal [d] to the ſquare of AC; therefore four times the rectangle AB, BC together with the ſquare of AC is equal to the gnomon AOH and the ſquare XH. but the gnomon AOH and XH make up the figure AEFD which is the ſquare of AD. therefore four times the rectangle AB, BC together with the ſquare of AC is equal to the ſquare of AD, that is, of AB and BC added together in one ſtraight line. Wherefore if a ſtraight line, &c. Q. E. D.



[b] 36. 1.

[c] 43. 1.

[d] Cor. 4. 2.

[c] 43. 1.

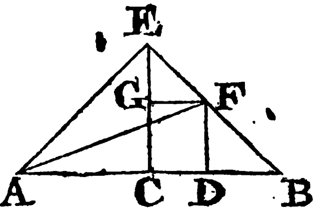
[d] Cor. 4. 2.

PROP. IX. THEOR.

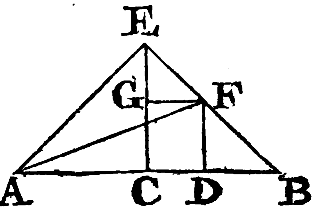
IF a ſtraight line be divided into two equal, and alſo into two unequal parts; the ſquares of the two unequal parts, are together double of the ſquare of half the line, and of the ſquare of the line between the points of ſection.

Let the ſtraight line AB be divided at the point C into two equal, and at D into two unequal parts. the ſquares of AD, DB are together double of the ſquares of AC, CD.

From the point C draw [a] CE at right angles to AB, and make it a. 11. 1. equal to AC or CB, and join EA, EB; thro' D draw [b] DF parallel b. 31. 1. to CE, and thro' F draw FG parallel to AB; and join AF. then becauſe AC is equal to CE, the angle EAC is equal [c] to the angle c. 5. 1. AEC; and becauſe the angle ACE is a right angle, the two others AEC, EAC together make one right angle [d]; and they are equal d. 32. 1. to one another; each of them therefore is half of a right angle. for the ſame reaſon each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle. and becauſe the angle GEF is half a right angle, and EGF a right angle, for it is equal [e] to the interior and oppoſite angle ECB, the remaining angle EFG is half a right angle; e. 29. 1. therefore the angle GEF is equal to the angle EFG, and the ſide EG equal [f] to the ſide GF. again, f. 6. 1. becauſe the angle at B is half a right angle, and FDB a right angle, for it is equal [e] to the interior and oppoſite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the ſide DF to [f] the ſide DB. and becauſe AC is equal to CE, the ſquare of AC is equal to the ſquare of CE; therefore the ſquares of AC, CE are double of the ſquare of AC. but the ſquare of EA is equal [g] to the ſquares of AC, CE, g. 47. 1. becauſe ACE is a right angle; therefore the ſquare of EA is double of the ſquare of AC. again, becauſe EG is equal to GF, the ſquare of EG is equal to the ſquare of GF; therefore the ſquares of EG, GF are double of the ſquare of GF; but the ſquare of EF is



equal to the ſquares of EG, GF; therefore the ſquare of EF is double of the ſquare GF. and GF is equal [h] to CD; therefore the
h. 34. 1. ſquare of EF is double of the ſquare of CD. but the ſquare of AE is likewiſe double of the ſquare of AC; therefore the ſquares of AE, EF are double of the ſquares of AC, CD. and the ſquare of
g. 47. 1. AF is equal [g] to the ſquares of AE, EF becauſe AEF is a right angle; therefore the ſquare of AF is double of the ſquares of AC, CD. but the ſquares of AD, DF are equal to the ſquare of AF, becauſe the angle ADF is a right angle; therefore the ſquares of AD, DF are double of the ſquares of AC, CD. and DF is equal to DB; therefore the ſquares of AD, DB are double of the ſquares of AC, CD. If therefore a ſtraight line, &c. Q. E. D.



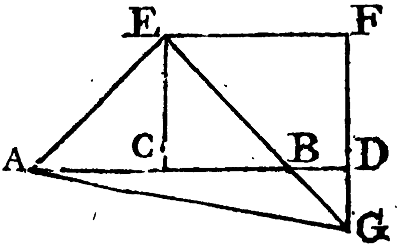
PROP. X. THEOR.

IF a ſtraight line be biſected, and produced to any point, the ſquare of the whole line thus produced, and the ſquare of the part of it produced are together double of the ſquare of half the line biſected, and of the ſquare of the line made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the ſquares of AD, DB are double of the ſquares of AC, CD.

a. 11. 1. From the point C draw [a] CE at right angles to AB, and make
b. 31. 1. it equal to AC, or CB, and join AE, EB; thro' E draw [b] EF parallel to AB, and thro' D draw DF parallel to CE. and becauſe the ſtraight line EF meets the parallels EC, FD, the angles CEF,
c. 29. 1. EFD are equal [c] to two right angles; and therefore the angles BEF, EFD are leſs than two right angles. but ſtraight lines which with another ſtraight line make the interior angles upon the ſame
d. 12. Ax. ſide leſs than two right angles, do meet [d] if produced far enough. therefore EB, FD ſhall meet, if produced, toward BD. let them meet in G, and join AG. then becauſe AC is equal to CE, the
e. 5. 1. angle CEA is equal [e] to the angle EAC; and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half a

right angle [*]. for the ſame reaſon, each of the angles CEB, EBC
is half a right angle; therefore AEB is a right angle. and becauſe * 32. 1.
EBC is half a right angle, DBG is alſo [f] half a right angle, for f. 15. 1.
they are vertically oppoſite; but BDG is a right angle, becauſe it
is equal [c] to the alternate angle DCE; therefore the remaining an- c. 29. 1.
gle DGB is half a right angle, and is therefore equal to the angle
DBG; wherefore alſo the ſide BD is equal [g] to the ſide DG. g. 6. 1.
again, becauſe EGF is half a
right angle, and that the an-
gle at F is a right angle, be-
cauſe it is equal [h] to the op- h. 34. 1.
poſite angle ECD, the re-
maining angle FEG is half a
right angle, and equal to the
angle EGF; wherefore alſo
the ſide GF is equal [g] to
the ſide FE. And becauſe EC is equal to CA, the ſquare of EC
is equal to the ſquare of CA; therefore the ſquares of EC, CA
are double of the ſquare of CA. but the ſquare of EA is equal [i] i. 47. 1.
to the ſquares of EC, CA; therefore the ſquare of EA is double
of the ſquare of AC. again, becauſe GF is equal to FE, the ſquare
of GF is equal to the ſquare of FE; and therefore the ſquares of
GF, FE are double of the ſquare of EF. but the ſquare of EG is
equal [i] to the ſquares of GF, FE; therefore the ſquare of EG is
double of the ſquare of EF. and EF is equal to CD, wherefore
the ſquare of EG is double of the ſquare of CD. but it was de-
monſtrated that the ſquare of EA is double of the ſquare of AC;
therefore the ſquares of AE, EG are double of the ſquares of AC,
CD. and the ſquare of AG is equal [i] to the ſquares of AE, EG; i. 47. 1.
therefore the ſquare of AG is double of the ſquares of AC, CD.
but the ſquares of AD, DG are equal [i] to the ſquare of AG;
therefore the ſquares of AD, DG are double of the ſquares of AC,
CD. but DG is equal to DB; therefore the ſquares of AD, DB
are double of the ſquares of AC, CD. Wherefore if a ſtraight
line, &c. Q. E. D.



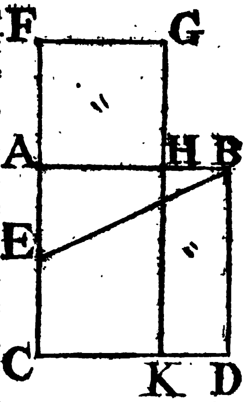
PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Let AB be the given ſtraight line; it is required to divide it into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

a. 46. 1. Upon AB deſcribe [a] the ſquare ABDC, biſect [b] AC in E, and b. 10. 1. join BE; produce CA to F, and make [c] EF equal to EB; and c. 3. 1. upon AF deſcribe [a] the ſquare FGHA, and produce GH to K. AB is divided in H ſo, that the rectangle AB, BH is equal to the ſquare of AH.

Becauſe the ſtraight line AC is biſected in E, and produced to the point F, the rectangle CF, FA, together with the ſquare of d. 6. 2. AE, is equal [d] to the ſquare of EF. but EF is equal to EB; therefore the rectangle CF, FA, together with the ſquare of AE is equal to the ſquare of e. 47. 1. EB. and the ſquares of BA, AE are equal [e] to the ſquare of EB, becauſe the angle EAB is a right angle; therefore the rectangle CF, FA, together with the ſquare of AE is equal to the ſquares of BA, AE. take away the ſquare of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the ſquare of AB. and the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the ſquare of AB; therefore FK is equal to AD. take away the common part AK, and the remainder FH is equal to the remainder HD. and HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the ſquare of AH. therefore the rectangle AB, BH is equal to the ſquare of AH. wherefore the ſtraight line AB is divided in H, ſo that the rectangle AB, BH is equal to the ſquare of AH. Which was to be done.

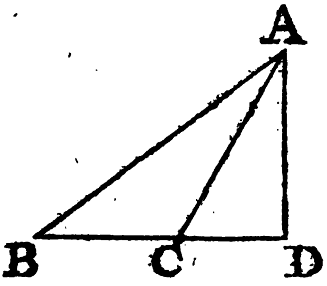


PROP. XII. THEOR.

IN obtuſe angled triangles, if a perpendicular be drawn from any of the acute angles to the oppoſite ſide produced, the ſquare of the ſide ſubtending the obtuſe angle, is greater than the ſquares of the ſides containing the obtuſe angle, by twice the rectangle contained by the ſide upon which when produced the perpendicular falls, and the ſtraight line intercepted without the triangle between the perpendicular and the obtuſe angle.

Let ABC be an obtuſe angled triangle, having the obtuſe angle
ACB, and from the point A let AD be drawn [a] perpendicular to a. 12. 1.
BC produced. the ſquare of AB is greater than the ſquares of AC,
CB by twice the rectangle BC, CD.

Becauſe the ſtraight line BD is divided into two parts in the
point C, the ſquare of BD is equal
[b] to the ſquares of BC, CD, and b. 4. 2.
twice the rectangle BC, CD. to
each of theſe equals add the ſquare
of DA; and the ſquares of BD, DA
are equal to the ſquares of BC, CD,
DA, and twice the rectangle BC,
CD. but the ſquare of BA is equal
[c] to the ſquares of BD, DA, be- c. 47. 1.
cauſe the angle at D is a right an-
gle; and the ſquare of CA is equal [c] to the ſquares of CD, DA.
therefore the ſquare of BA is equal to the ſquares of BC, CA,
and twice the rectangle BC, CD; that is, the ſquare of BA is
greater than the ſquares of BC, CA, by twice the rectangle BC,
CD. Therefore in obtuſe angled triangles, &c. Q. E. D.

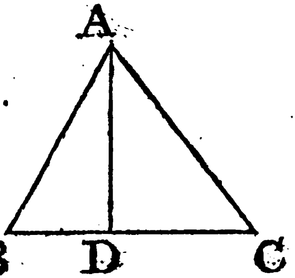


PROP. XIII. THEOR.

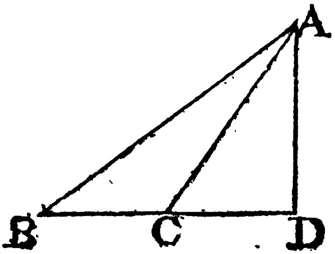
See N. IN every triangle the ſquare of the ſide ſubtending any of the acute angles, is leſs than the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the ſtraight line intercepted between the perpendicular let fall upon it from the oppoſite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC one of the ſides containing it let fall the
a. 12. 1. perpendicular [a] AD from the oppoſite angle. the ſquare of AC oppoſite to the angle B, is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.

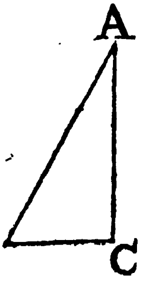
First, Let AD fall within the triangle ABC; and becauſe the ſtraight line CB is divided into two parts in the point D, the ſquares of
b. 7. 2. CB, BD are equal [b] to twice the rectangle contained by CB, BD, and the ſquare of DC. to each of theſe equals add the ſquare of AD, therefore the ſquares of CB, BD, DA are equal to twice the rectangle CB, BD, and the ſquares of AD,
c. 47. 1. DC. but the ſquare of AB is equal [c] to the ſquares of BD, DA, becauſe the angle BDA is a right angle; and the ſquare of AC is equal to the ſquares of AD, DC, therefore the ſquares of CB, BA are equal to the ſquare of AC, and twice the rectangle CB, BD; that is, the ſquare of AC alone is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.



Secondly, Let AD fall without the triangle ABC. then becauſe the angle at D is a right angle, the
d. 16. 1. angle ACB is greater [d] than a right angle; and therefore the ſquare of
e. 12. 2. AB is equal [e] to the ſquares of AC, CB and twice the rectangle BC, CD. to theſe equals add the ſquare of BC, and the ſquares of AB, BC



are equal to the ſquare of AC, and twice the ſquare of BC, and twice the rectangle BC, CD. but becauſe BD is divided into two parts in C, the rectangle DB, BC is equal [f] to the rectangle BC, CD and the ſquare of BC. and the doubles of theſe are equal. therefore the ſquares of AB, BC are equal to the ſquare of AC, and twice the rectangle DB, BC. therefore the ſquare of AC alone, is leſs than the ſquares of AB, BC, by twice the rectangle DB, BC. f. 3. 2.



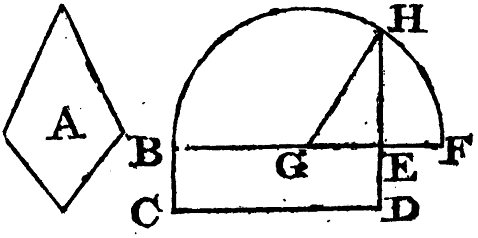
Laſtly, let the ſide AC be perpendicular to BC; then is BC the ſtraight line between the perpendicular and the acute angle at B. and it is manifeſt that the ſquares of AB, BC are equal [c] to the ſquare of AC, and twice the ſquare of BC. Therefore in every triangle, &c. Q. E. D. c. 47. 1.

PROP. XIV. PROB.

TO deſcribe a ſquare that ſhall be equal to a given rectilineal figure. See N.

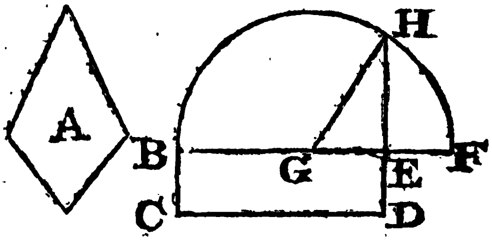
Let A be the given rectilineal figure; it is required to deſcribe a ſquare that ſhall be equal to A.

Deſcribe [a] the rectangular parallelogram BCDE equal to the rectilineal figure A. If then the ſides of it BE, ED are equal to one another, it is a ſquare, and what was required is now done. but if they are not equal, produce one of them BE to F, and make EF equal to ED, and biſect BF in G; and from the center G, at the diſtance GB or GF deſcribe the ſemicircle BHF, and produce DE to H, and join GH. therefore becauſe the ſtraight line BF is divided into two equal parts in the point G, and into two unequal at E, the rectangle BE, EF, together with the ſquare of EG, is equal [b] to the ſquare of GF. but GF is equal to GH; therefore the rectangle BE, EF, together with the ſquare of EG, is equal to the ſquare of GH. but the ſquares a. 45. 1. b. 5. 2.



of HE, EG are equal [c] to the ſquare of GH. therefore the rectangle BE, EF together with the ſquare of EG is equal to the ſquares of HE, EG. take away the ſquare of EG, which is common to both; and the remaining rectangle BE, EF is equal to the ſquare of EH. but the rectangle contained by BE, EF is the parallelogram BD, becauſe EF is equal to ED; therefore BD is equal to the ſquare of EH. but BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the ſquare of EH. wherefore a ſquare has been made equal to the given rectilineal figure A, viz. the ſquare deſcribed upon EH. Which was to be done.

c. 47. 1.



THE ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

I.

EQUAL circles are thoſe of which the diameters are equal, or from the centers of which the ſtraight lines to the circumferences are equal.

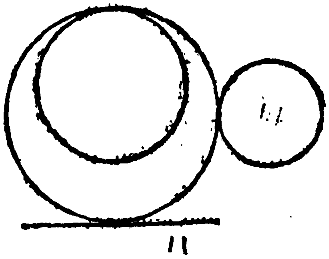
'This is not a Definition but a Theorem, the truth of which 'is evident; for if the circles be applied to one another, ſo that 'their centers coincide, the circles muſt likewiſe coincide, ſince 'the ſtraight lines from their centers are equal.'

II.

A ſtraight line is ſaid to touch a circle, when it meets the circle and being produced does not cut it.

III.

Circles are ſaid to touch one another, which meet but do not cut one another.

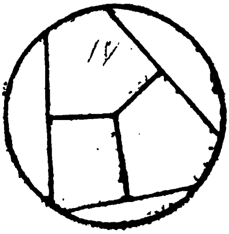


IV.

Straight lines are ſaid to be equally diſtant from the center of a circle, when the perpendiculars drawn to them from the center are equal.

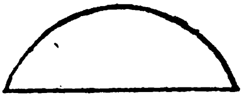
V.

And the ſtraight line on which the greater perpendicular falls, is ſaid to be farther from the center.



VI.

A ſegment of a circle is the figure contained by a ſtraight line and the circumference it cuts off.

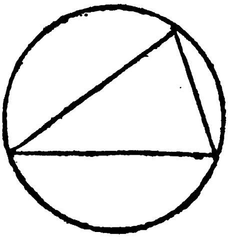


VII.

" The angle of a ſegment is that which is contained by the ſtraight
" line and the circumference."

VIII.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the circumference of the ſegment, to the extremities of the ſtraight line which is the baſe of the ſegment.

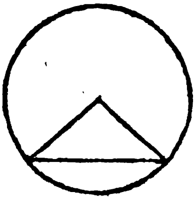


IX.

And an angle is ſaid to inſiſt or ſtand upon the circumference intercepted between the ſtraight lines that contain the angle.

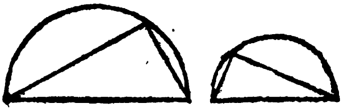
X.

The ſector of a circle is the figure contained by two ſtraight lines drawn from the center, and the circumference between them.



XI.

Similar ſegments of a circle, are thoſe in which the angles are equal, or which contain equal angles.

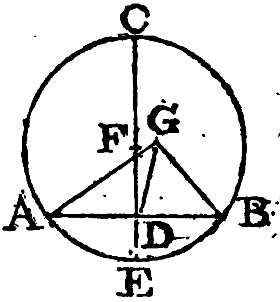


PROP. I. PROB.

To find the center of a given circle.

Let ABC be the given circle; it is required to find its center.
a. 10. 1. Draw within it any ſtraight line AB, and biſect [a] it in D; from
b. 11. 1. the point D draw [b] DC at right angles to AB, and produce it to E, and biſect CE in F. the point F is the center of the circle ABC.

For if it be not, let, if poſſible, G be the center, and join GA, GD, GB. then becauſe DA is equal to DB, and DG common to the two triangles ADG, BDG, the two ſides AD, DG are equal to the two BD, DG, each to each; and the baſe GA is equal to the baſe GB, becauſe they are drawn from the center G *. therefore the angle ADG is equal [c] to the angle GDB. but when a ſtraight line ſtanding upon another ſtraight line, makes the adjacent angles equal to one another, each of the angles is a right angle [d]. therefore the angle GDB is a right angle. but FDB is likewiſe a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the leſs, which is impoſſible. therefore G is not the center of the circle ABC. in the ſame manner it can be ſhewn, that no other point but F is the center; that is, F is the center of the circle ABC. Which was to be found.



c. 8. 1.

d. 10. Def. 1.

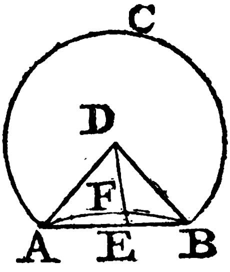
Cor. From this it is manifeſt, that if in a circle a ſtraight line biſect another at right angles, the center of the circle is in the line which biſects the other.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the ſtraight line which joins them ſhall fall within the circle.

Let ABC be a circle; and A, B any two points in the circumference; the ſtraight line drawn from A to B ſhall fall within the circle.

For if it do not, let it fall, if poſſible, without, as AEB; find [a] D the center of the circle ABC, and join AD, DB, and produce DF any ſtraight line meeting the circumference AB, to E. then becauſe DA is equal to DB, the angle DAB is equal [b] to the angle DBA; and becauſe



a. 1. 3.

b. 5. 1.

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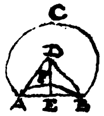
* N. B. Whenever the expreſſion "ſtraight lines from the center" or "drawn from the center" occurs, it is to be underſtood that they are drawn to the circumference.



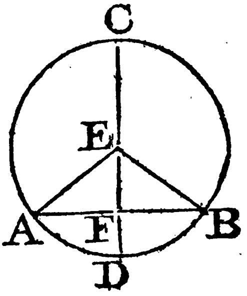
AE a ſide of the triangle DAE is produced to B, the angle DEB is greater [c] than the angle DAE; but DAE is equal to the angle
c. 16. 1. DBE, therefore the angle DEB is greater than the angle DBE.
d. 19. 1. but to the greater angle the greater ſide is oppoſite [d]; DB is therefore greater than DE. but DB is equal to DF; wherefore DF is greater than DE, the leſs than the greater, which is impoſſible. therefore the ſtraight line drawn from A to B does not fall without the circle. in the ſame manner, it may be demonſtrated that it does not fall upon the circumference. it falls therefore within it. Wherefore if any two points, &c. Q. E. D.

PROP. III. THEOR.

IF a ſtraight line drawn thro' the center of a circle, biſect a ſtraight line in it which does not paſs thro' the center, it ſhall cut it at right angles. and if it cuts it at right angles, it ſhall biſect it.

Let ABC be a circle; and let CD a ſtraight line drawn thro' the center biſect any ſtraight line AB, which does not paſs thro' the center, in the point F. it cuts it alſo at right angles.

a. 1. 3. Take [a] E the center of the circle, and join EA, EB. then becauſe AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two ſides in the one equal to two ſides in the other. and the baſe EA is equal to the baſe EB; therefore the angle AFE is equal
b. 8. 1. [b] to the angle BFE. but when a ſtraight line ſtanding upon another makes the adjacent angles equal to one another, each
c. 10. Def. 1. of them is a right [c] angle. therefore each of the angles AFE, BFE is a right angle; wherefore the ſtraight line CD drawn thro' the center biſecting another AB that does not paſs thro' the center, cuts the ſame at right angles.



But let CD cut AB at right angles; CD alſo biſects it, that is, AF is equal to FB.

The ſame conſtruction being made, becauſe EA, EB from the
d. 5. 1. center are equal to one another, the angle EAF is equal [d] to the angle EBF; and the right angle AFE is equal to the right angle BFE. therefore in the two triangles EAF, EBF there are two an-

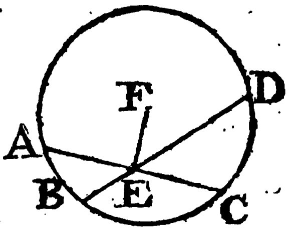
gles in one equal to two angles in the other, and the ſide EF which is oppoſite to one of the equal angles in each, is common to both; therefore the other ſides are equal [c], AF therefore is equal to FB. Wherefore if a ſtraight line, &c. Q. E. D.

c. 26. 1.

PROP. IV. THEOR.

IF in a circle two ſtraight lines cut one another which do not both paſs thro' the center, they do not biſect each the other.

Let ABCD be a circle, and AC, BD two ſtraight lines in it which cut one another in the point E, and do not both paſs thro' the center. AC, BD do not biſect one another.



For, if it is poſſible, let AE be equal to EC, and BE to ED. if one of the lines paſs thro' the center, it is plain that it cannot be biſected by the other which does not paſs thro' the center. but if neither of them paſs thro' the center, take [a] F the center of the circle, and join EF. and becauſe FE a ſtraight line thro' the center, biſects another AC which does not paſs thro' the center, it ſhall cut it at right [b] angles; wherefore FEA is a right angle. again, becauſe the ſtraight line FE biſects the ſtraight line BD which does not paſs thro' the center, it ſhall cut it at right [b] angles; wherefore FEB is a right angle. and FEA was ſhewn to be a right angle; therefore FEA is equal to the angle FEB, the leſs to the greater, which is impoſſible. therefore AC, BD do not biſect one another. Wherefore if in a circle, &c. Q. E. D.

a. 1. 3.

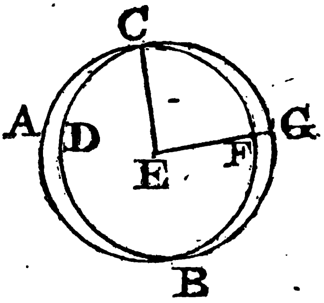
b. 3. 3.

PROP. V. THEOR.

IF two circles cut one another, they ſhall not have the ſame center.

Let the two circles ABC, CDG cut one another in the points B, C; they have not the ſame center.

For, if it be poſſible, let E be their center; join EC, and draw any ſtraight line EFG meeting them in F and G. and becauſe E is the center of the circle ABC, CE is equal to EF. again, becauſe E is the center of the circle CDG, CE is equal to EG. but CE was ſhewn to be equal to EF; therefore EF is equal to EG, the leſs to the greater, which is impoſſible. therefore E is not the center of the circles ABC, CDG. Wherefore if two circles, &c. Q. E. D.

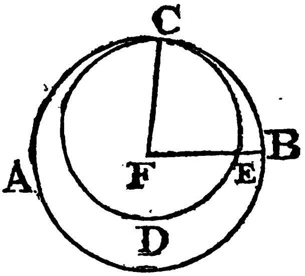


PROP. VI. THEOR.

IF two circles touch one another internally, they ſhall not have the ſame center.

Let the two circles ABC, CDE touch one another internally in the point C. they have not the ſame center.

For if they can, let it be F; join FC, and draw any ſtraight line FEB meeting them in E and B. and becauſe F is the center of the circle ABC, CF is equal to FB. alſo becauſe F is the center of the circle CDE, CF is equal to FE. and CF was ſhewn equal to FB; therefore FE is equal to FB, the leſs to the greater, which is impoſſible. wherefore F is not the center of the circles ABC, CDE. Therefore if two circles, &c. Q. E. D.



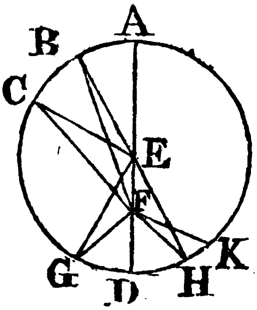
PROP. VII. THEOR.

IF any point be taken in the diameter of a circle which is not the center, of all the ſtraight lines which can be drawn from it to the circumference, the greateſt is that in which the center is, and the other part of that diameter is the leaſt; and of any others, that which is nearer to the line which paſſes thro' the center is always greater than one more remote. and from the ſame point there can be drawn only two ſtraight lines that are equal to one another, one upon each ſide of the ſhorteſt line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the center. let the center be E; of all the ſtraight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greateſt, and FD the other part of the diameter AD is the leaſt; and of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE; and becauſe two ſides of a triangle are greater [a] than the third, BE, EF are greater than BF; but AE is equal to EB, therefore AE, EF, that is AF, is greater than BF. again, becauſe BE is equal to CE, and FE common to the triangles BEF, CEF; the two ſides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF, therefore the baſe BF is greater [b] than the baſe FC. for the ſame reaſon, CF is greater than GF. again, becauſe GF, FE are greater [a] than EG, and EG is equal to ED; GF, FE are greater than ED. take away the common part FE, and the remainder GF is greater than the remainder FD. therefore FA is the greateſt, and FD the leaſt of all the ſtraight lines from F to the circumference; and BF is greater than CF, and CF than GF.

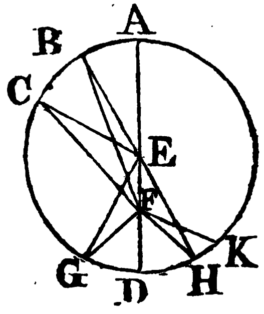
a. 20. 1.



b. 24. 1.

Alſo there can be drawn only two equal ſtraight lines from the point F to the circumference, one upon each ſide of the ſhorteſt line

Book III. FD. at the point E in the ſtraight line EF, make [c] the angle FEH equal to the angle GEF, and join FH. c. 23. 1. then becauſe GE is equal to EH, and EF common to the two triangles GEF, HEF; the two ſides GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF, d. 4. 1. therefore the baſe FG is equal [d] to the baſe FH. but beſides FH no other ſtraight line can be drawn from F to the circumference equal to FG. for if there can, let it be FK, and becauſe FK is equal to FG, and FG to FH, FK is equal to FH, that is, a line nearer to that which paſſes thro' the center is equal to one which is more remote; which is impoſſible. Therefore if any point be taken, &c. Q. E. D.

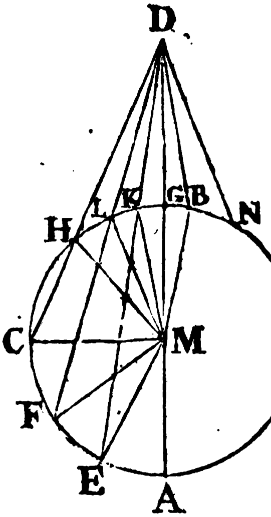


PROP. VIII. THEOR.

IF any point be taken without a circle, and ſtraight lines be drawn from it to the circumference, whereof one paſſes thro' the center; of thoſe which fall upon the concave circumference the greateſt is that which paſſes thro' the center; and of the reſt, that which is nearer to that thro' the center is always greater than the more remote. but of thoſe which fall upon the convex circumference, the leaſt is that between the point without the circle, and the diameter; and of the reſt, that which is nearer to the leaſt is always leſs than the more remote. and only two equal ſtraight lines can be drawn from the point unto the circumference, one upon each ſide of the leaſt.

Let ABC be a circle, and D any point without it, from which let the ſtraight lines DA, DE, DF, DC be drawn to the circumference, whereof DA paſſes thro' the center. of thoſe which fall upon the concave part of the circumference AEFC, the greateſt is AD which paſſes thro' the center; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC. but of thoſe which fall upon the convex circumference HLKG, the leaſt is DG between the point D and the diameter AG; and the nearer to it is always leſs than the more remote, viz. DK than DL, and DL than DH.

Take [a] M the center of the circle ABC, and join ME, MF, MC, MK, ML, MH. and becauſe AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater [b] than ED, therefore alſo AD is greater than ED. again, becauſe ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is greater than the angle FMD, therefore the baſe ED is greater [c] than the baſe FD. in like manner it may be ſhewn that FD is greater than CD. therefore DA is the greateſt; and DE greater than DF, and DF than DC. and becauſe MK, KD are greater [b] than MD, and MK is equal to MG, the remainder KD is greater [d] than the remainder GD, that is, GD is leſs than KD. and becauſe MK, DK are drawn to the point K within the triangle MLD from M, D the extremities of its ſide MD; MK, KD are leſs [e] than ML, LD, whereof MK is equal to ML, therefore the remainder DK is leſs than the remainder DL. in like manner it may be ſhewn that DL is leſs than DH. therefore DG is the leaſt, and DK leſs than DL, and DL than DH. Alſo there can be drawn only two equal ſtraight lines from the point D to the circumference, one upon each ſide of the leaſt. at the point M in the ſtraight line MD, make the angle DMB equal to the angle DMK, and join DB. and becauſe MK is equal to MB, and MD common to the triangles KMD, BMD, the two ſides KM, MD are equal to the two BM, MD; and the angle KMD is equal to the angle BMD, therefore the baſe DK is equal [f] to the baſe DB. but beſides DB there can be no ſtraight line drawn from D to the circumference equal to DK. for if there can, let it be DN; and becauſe DK is equal to DN, and alſo to DB, therefore DB is equal to DN, that is the nearer to the leaſt equal to the more remote, which is impoſſible. If therefore any point, &c. Q. E. D.

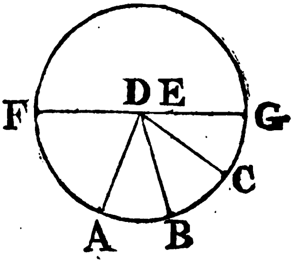


a. 1. 3.
b. 20. 1.
c. 24. 1.
d. 4. Ax.
e. 21. 1.
f. 4. 1.

PROP. IX. THEOR.

IF a point be taken within a circle, from which there fall more than two equal ſtraight lines to the circumference, that point is the center of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ſtraight lines, viz. DA, DB, DC. the point D is the center of the circle.

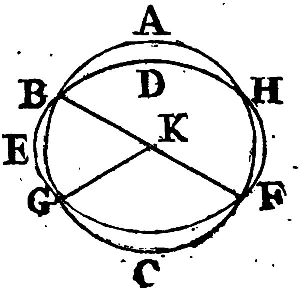


For if not, let E be the center, join DE and produce it to the circumference in F, G; then FG is a diameter of the circle ABC. and becauſe in FG the diameter of the circle ABC there is taken the point D which is not the center, DG ſhall be the greateſt line from it to the circumference, and DC greater [a] than DB, and DB than DA. but they are likewiſe equal, which is impoſſible. therefore E is not the center of the circle ABC. in like manner it may be demonſtrated that no other point but D is the center; D therefore is the center. Wherefore if a point be taken, &c. Q. E. D.

[a] 7. 3.

PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.



If it be poſſible, let the circumference ABC cut the circumference DEF in more than two points, viz. in B, G, F; take the center K of the circle ABC, and join KB, KG, KF. and becauſe within the circle DEF there is taken the point K from which to the circumference DEF fall more than two equal ſtraight lines KB, KG, KF, the point K is [a] the center of the

[a] 9. 3.

circle DEF. but K is alſo the center of the circle ABC; therefore the ſame point is the center of two circles that cut one another; which is impoſſible [b]. Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

b. 5. 3.

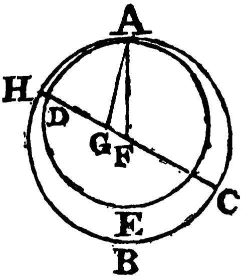
PROP. XI. THEOR.

IF two circles touch each other internally, the ſtraight line which joins their centers being produced ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch each other internally in the point A, and let F be the center of the circle ABC, and G the center of the circle ADE. the ſtraight line which joins the centers F, G being produced paſſes thro' the point A.

For if not, let it fall otherwiſe, if poſſible, as FGDH, and join AF, AG. and becauſe AG, GF are greater [a] than FA, that is than FH, for FA is equal to FH, both being from the ſame center; take away the common part FG, therefore the remainder AG is greater than the remainder GH. but AG is equal to GD, therefore GD is greater than GH, the leſs than the greater, which is impoſſible. therefore the ſtraight line which joins the points F, G cannot fall otherwiſe than upon the point A, that is, it muſt paſs thro' it. Therefore if two circles, &c. Q. E. D.

a. 20. 1.



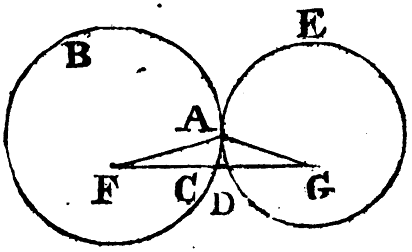
PROP. XII. THEOR.

IF two circles touch each other externally, the ſtraight line which joins their centers ſhall paſs thro' the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the center of the circle ABC, and G the center of ADE. the ſtraight line which joins the points F, G ſhall paſs thro' the point of contact A.

For if not, let it paſs otherwiſe, if poſſible, as FCDG, and join

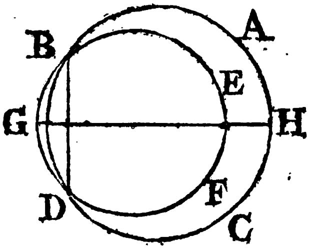
Book III. FA, AG. and becauſe F is the center of the circle ABC, AF is equal to FC. alſo becauſe G is the center of the circle ADE, AG is equal to GD. therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG.
a. 20. 1. but it is alſo leſs [a]; which is impoſſible. therefore the ſtraight line which joins the points F, G ſhall not paſs otherwiſe than thro' the point of contact A, that is, it muſt paſs thro' it. Therefore if two circles, &c. Q. E. D.

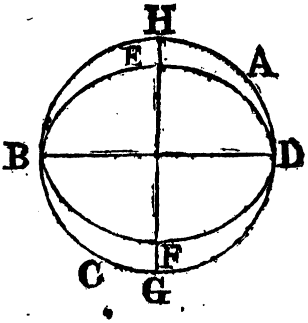


PROP. XIII. THEOR.

See N. **ONE circle cannot touch another in more points than one, whether it touches it on the inſide or outſide.**

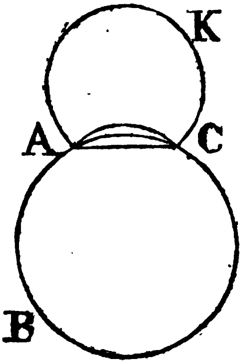
For, if it be poſſible, let the circle EBF touch the circle ABC in more points than one, and firſt on the inſide, in the points B, D;
a. 10.11. 1. join BD, and draw [a] GH biſecting BD at right angles. therefore becauſe the points B, D are in the circumference of each of the





b. 2. 3. circles, the ſtraight line BD falls within [b] each of them. and their
c. Cor. 1. 3. centers are [c] in the ſtraight line GH which biſects BD at right
d. 11. 3. angles; therefore GH paſſes thro' the point of contact [d]. but it does not paſs thro' it, becauſe the points B, D are without the ſtraight line GH, which is abſurd. therefore one circle cannot touch another on the inſide in more points than one.

Nor can two circles touch one another on the outſide in more than one point. for, if it be poſſible, let the circle ACK touch the circle ABC in the points A, C, and join AC. therefore becauſe the two points A, C are in the circumference of the circle ACK, the ſtraight line AC which joins them ſhall fall within [b] the circle ACK. b. 2. 3. and the circle ACK is without the circle ABC, and therefore the ſtraight line AC is without this laſt circle; but becauſe the points A, C, are in the circumference of the circle ABC, the ſtraight line AC muſt be within [b] the ſame circle, which is abſurd. therefore one circle cannot touch another on the outſide in more than one point. and it has been ſhewn that they cannot touch on the inſide in more points than one. therefore one circle, &c. Q. E. D.

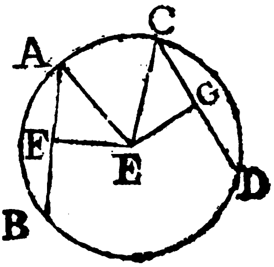


PROP. XIV. THEOR.

EQUAL ſtraight lines in a circle are equally diſtant from the center; and thoſe which are equally diſtant from the center, are equal to one another.

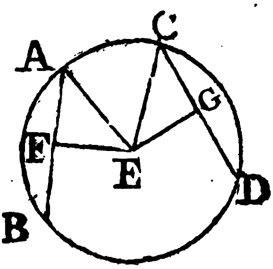
Let the ſtraight lines AB, CD in the circle ABDC be equal to one another; they are equally diſtant from the center.

Take E the center of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD. then becauſe the ſtraight line EF paſſing thro' the center cuts the ſtraight line AB, which does not paſs thro' the center, at right angles, it alſo biſects [a] it. a. 3. 3. wherefore AF is equal to FB, and AB double of AF. for the ſame reaſon CD is double of CG. and AB is equal to CD, therefore AF is equal to CG. and becauſe AE is equal to EC, the ſquare of AE is equal to the ſquare of EC. but the ſquares of AF, FE are equal [b] to the ſquare of AE, becauſe the b. 47. 1. angle AFE is a right angle; and for the like reaſon the ſquares of EG, GC are equal to the ſquare of EC. therefore the ſquares of AF, FE are equal to the ſquares of CG,



Book III. GE, of which the ſquare of AF is equal to the ſquare of CG, becauſe AF is equal to CG; therefore the remaining ſquare of FE is equal to the remaining ſquare of EG, and the ſtraight line FE is therefore equal to EG. but ſtraight lines in a circle are ſaid to be equally diſtant from the center, when the perpendiculars drawn to them from the center are equal [c]. therefore AB, CD are equally diſtant from the center.

c. 4. Def. 3.



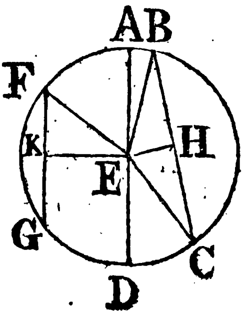
Next, if the ſtraight lines AB, CD be equally diſtant from the center, that is, if FE be equal to EG; AB is equal to CD. for, the ſame conſtruction being made, it may, as before, be demonſtrated that AB is double of AF and CD double of CG, and that the ſquares of EF, FA are equal to the ſquares of EG, GC; of which the ſquare of FE is equal to the ſquare of EG, becauſe FE is equal to EG; therefore the remaining ſquare of AF is equal to the remaining ſquare of CG; and the ſtraight line AF is therefore equal to CG. and AB is double of AF, and CD double of CG; wherefore AB is equal to CD. Therefore equal ſtraight lines, &c. Q. E. D.

PROP. XV. THEOR.

See N.

THE diameter is the greateſt ſtraight line in a circle; and of all others, that which is nearer to the center is always greater than one more remote; and the greater is nearer to the center than the leſs.

Let ABCD be a circle, of which the diameter is AD, and center E; and let BC be nearer to the center than FG. AD is greater than any ſtraight line BC which is not a diameter, and BC greater than FG.



From the center draw EH, EK perpendiculars to BC, FG, and join EB, EC, EF; and becauſe AE is equal to EB, and ED to EC, AD is equal to EB, EC. but EB, EC, are greater [a] than BC, wherefore alſo AD is greater than BC.

a. 20. 1.

And becauſe BC is nearer to the center than FG, EH is leſs [b] than EK. but, as was demonſtrated in the preceding, BC is double of BH, and FG double of FK, and the ſquares of EH, HB are equal to the ſquares of EK, KF, of which the ſquare of EH is leſs than the ſquare of EK, becauſe EH is leſs than EK; therefore the ſquare of BH is greater than the ſquare of FK, and the ſtraight line BH greater than FK; and therefore BC is greater than FG.

b. 5. Def. 3.

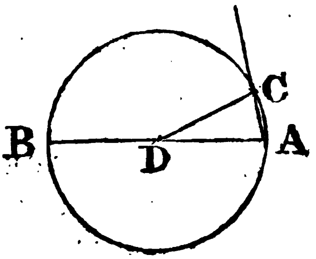
Next, let BC be greater than FG; BC is nearer to the center than FG, that is, the ſame conſtruction being made, EH is leſs than EK. becauſe BC is greater than FG, BH likewiſe is greater than FK. and the ſquares of BH, HE are equal to the ſquares of FK, KE, of which the ſquare of BH is greater than the ſquare of FK, becauſe BH is greater than FK; therefore the ſquare of EH is leſs than the ſquare of EK, and the ſtraight line EH leſs than EK. Wherefore the diameter, &c. Q. E. D.

PROP. XVI. THEOR.

THE ſtraight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no ſtraight line can be drawn between that ſtraight line and the circumference from the extremity, ſo as not to cut the circle; or, which is the ſame thing, no ſtraight line can make ſo great an acute angle with the diameter at its extremity, or ſo ſmall an angle with the ſtraight line which is at right angles to it, as not to cut the circle.

See N.

Let ABC be a circle the center of which is D, and the diameter AB; the ſtraight line drawn at right angles to AB from its extremity A, ſhall fall without the circle.



For if it does not, let it fall, if poſſible, within the circle as AC, and draw DC to the point C where it meets the circumference. and becauſe DA is equal to DC, the angle DAC is equal [a] to the angle ACD; but DAC is a right angle, therefore ACD is a right angle, and

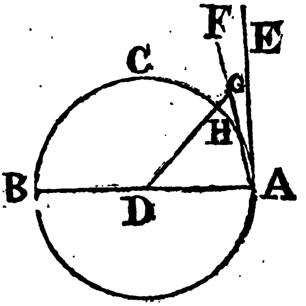
a. 5. 1.

the angles DAC, ACD are therefore equal to two right angles; which is impoſſible [b]. therefore the ſtraight line drawn from A at right angles to BA does not fall within the circle. in the ſame manner it may be demonſtrated that it does not fall upon the circumference; therefore it muſt fall without the circle, as AE.

b. 17. 1.

And between the ſtraight line AE and the circumference no ſtraight line can be drawn from the point A which does not cut the circle. for, if poſſible, let FA be between them, and from the point D draw [c] DG perpendicular to FA, and let it meet the circumference in H. and becauſe AGD is a right angle, and DAG leſs [b] than a right angle, DA is greater [d] than DG. but DA is equal to DH; therefore DH is greater than DG, the leſs than the greater, which is impoſſible. therefore no ſtraight line can be drawn from the point A between AE and the circumference, which does not cut the circle. or, which amounts to the ſame thing, however great an acute angle a ſtraight line makes with the diameter at the point A, or however ſmall an angle it makes with AE, the circumference paſſes between that ſtraight line and the perpendicular AE. ‘ And this is all that is to be underſtood, ‘ when in the Greek text and tranſlations from it, the angle of the ‘ ſemicircle is ſaid to be greater than any acute rectilineal angle, ‘ and the remaining angle leſs than any rectilineal angle.’

c. 12. 1.
d. 19. 1.



Cor. From this it is manifeſt that the ſtraight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, becauſe if it did meet the circle in two, it would fall within it [e]. ‘ Alſo it is evident that there can be but one ſtraight line ‘ which touches the circle in the ſame point.’

e. 2. 3.

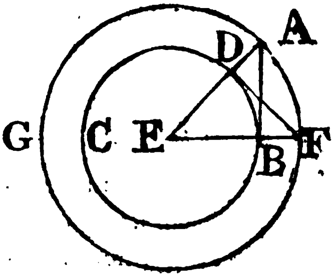
PROP. XVII. PROB.

TO draw a ſtraight line from a given point, either without or in the circumference, which ſhall touch a given circle.

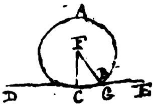
Firſt, Let A be a given point without the given circle BCD; it is required to draw a ſtraight line from A which ſhall touch the circle.

Find [a] the center E of the circle, and join AE; and from the a. 1. 3.
center E, at the diſtance EA deſcribe the circle AFG; from the
point D draw [b] DF at right angles to EA, and join EBF, AB. b. 11. 1.
AB touches the circle BCD.

Becauſe E is the center of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two ſides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the baſe DF is equal to the baſe AB, and the triangle FED to the triangle AEB, and the other
angles to the other angles [c]. c. 4. 1.
therefore the angle EBA is equal to the angle EDF. but EDF is a right angle, wherefore EBA is a right angle. and EB is drawn from the center; but a ſtraight line drawn from the extremity of
a diameter, at right angles to it, touches the circle [d]. therefore AB d. Cor. 16. 3.
touches the circle; and it is drawn from the given point A. Which was to be done.



But if the given point be in the circumference of the circle, as the point D, draw DE to the center E, and DF at right angles to DE; DF touches the circle [d].



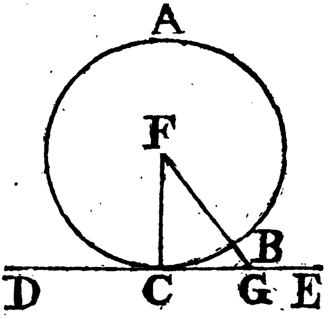
PROP. XVIII. THEOR.

IF a ſtraight line touches a circle, the ſtraight line drawn from the center to the point of contact, ſhall be perpendicular to the line touching the circle.

Let the ſtraight line DE touch the circle ABC in the point C, take the center F, and draw the ſtraight line FC; FC is perpendicular to DE.

For if it be not, from the point F draw FBG perpendicular to
DE; and becauſe FGC is a right angle, GCF is [b] an acute angle; b. 17. 1.
and to the greater angle the greateſt [c] ſide is oppoſite. therefore FC c. 19. 1.

Book III. is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the leſs than the greater, which is impoſſible. wherefore FG is not perpendicular to DE. in the ſame manner it may be ſhewn, that no other is perpendicular to it beſides FC, that is, FC is perpendicular to DE. Therefore if a ſtraight line, &c. Q. E. D.

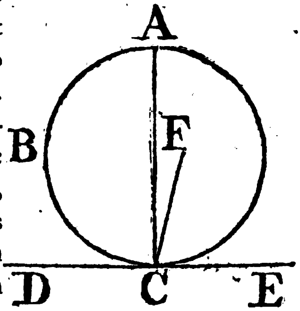


PROP. XIX. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn at right angles to the touching line, the center of the circle ſhall be in that line.

Let the ſtraight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the center of the circle is in CA.

For if not, let F be the center, if poſſible, and join CF. Becauſe DE touches the circle ABC, and FC is drawn from the center to the point of contact, FC is perpendicular [a] to DE; therefore FCE is a right angle. but ACE is alſo a right angle; therefore the angle FCE is equal to the angle ACE, the leſs to the greater, which is impoſſible. wherefore F is not the center of the circle ABC. in the ſame manner it may be ſhewn that no other point which is not in CA, is the center; that is, the center is in CA. Therefore if a ſtraight line, &c. Q. E. D.



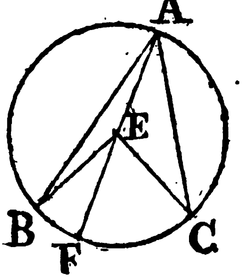
[a] 18. 3.

PROP. XX. THEOR.

See N.

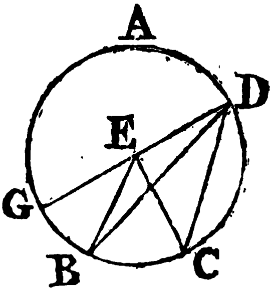
THE angle at the center of a circle is double of the angle at the circumference, upon the ſame baſe, that is, upon the ſame part of the circumference.

Let ABC be a circle, and BEC an angle at the center, and BAC an angle at the circumference, which have the ſame circumference BC for their baſe; the angle BEC is double of the angle BAC.



Firſt, Let E the center of the circle be within the angle BAC, and join AE, and produce it to F. Becauſe EA is equal to EB, the angle EAB is equal [a] to the angle EBA; therefore the angles EAB, EBA are double of the angle EAB; but the angle BEF is equal [b] to the angles EAB, EBA; therefore alſo the angle BEF is double of the angle EAB. for the ſame reaſon, the angle FEC is double of the angle EAC. therefore the whole angle BEC is double of the whole angle BAC.

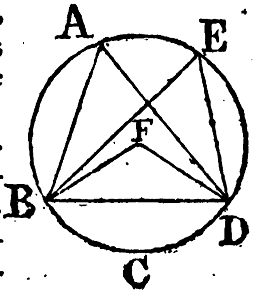
a. 5. 1.
b. 32. 1.



Again, Let BDC be inflected to the circumference, ſo that E the center of the circle be without the angle BDC, and join DE and produce it to G. It may be demonſtrated, as in the firſt caſe, that the angle GEC is double of the angle GDC, and that GEB a part of the firſt is double of GDB a part of the other; therefore the remaining angle BEC is double of the remaining angle BDC. Therefore the angle at the center, &c. Q.E.D.

PROP. XXI. THEOR.

THE angles in the ſame ſegment of a circle are equal to one another. See N.



Let ABCD be a circle, and BAD, BED angles in the ſame ſegment BAED; the angles BAD, BED are equal to one another.

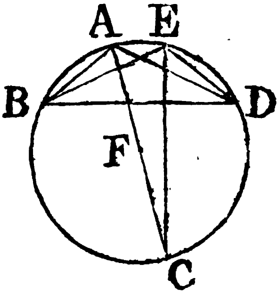
Take F the center of the circle ABCD. and, firſt, let the ſegment BAED be greater than a ſemicircle, and join BF, FD. and becauſe the angle BFD is at the center, and the angle BAD at the circumfe-

Book III.

a. 20. 3.

rence, and that they have the ſame part of the circumference, viz. BCD for their baſe, therefore the angle BFD is double [a] of the angle BAD. for the ſame reaſon, the angle BFD is double of the angle BED. therefore the angle BAD is equal to the angle BED.

But if the ſegment BAED be not greater than a ſemicircle, let BAD, BED be angles in it; theſe alſo are equal to one another. draw AF to the center, and produce it to C, and join CE. therefore the ſegment BADC is greater than a ſemicircle; and the angles in it BAC, BEC are equal, by the firſt caſe. for the ſame reaſon, the angles CAD, CED are equal. therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the ſame ſegment, &c. Q. E. D.



PROP. XXII. THEOR.

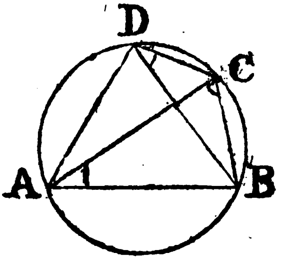
THE oppoſite angles of any quadrilateral figure deſcribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its oppoſite angles are together equal to two right angles.

a. 32. 1.

b. 21. 3.

Join AC, BD; and becauſe the three angles of every triangle are equal [a] to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles. but the angle CAB is equal [b] to the angle CDB, becauſe they are in the ſame ſegment BADC; and the angle ACB is equal to the angle ADB, becauſe they are in the ſame ſegment ADCB. therefore the whole angle ADC is equal to the angles CAB, ACB. to each of theſe equals add the angle ABC, therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC. but ABC, CAB, BCA are equal to two right angles; therefore alſo the angles ABC, ADC are equal to two right angles. in the ſame manner the angles BAD, DCB

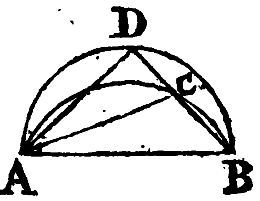


may be ſhewn to be equal to two right angles. Therefore the oppoſite angles, &c. Q. E. D.

PROP. XXIII. THEOR.

UPON the ſame ſtraight line, and upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles, not coinciding with one another. See N.

If it be poſſible, let the two ſimilar ſegments of circles, viz. ACB, ADB be upon the ſame ſide of the ſame ſtraight line AB, not coinciding with one another. then becauſe the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point [a]. one of the ſegments muſt therefore fall within the other; let ACB fall within ADB, and draw the ſtraight line BCD, and join CA, DA. and becauſe the ſegment ACB is ſimilar to the ſegment ADB, and that ſimilar ſegments of circles contain [b] equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impoſſible [c]. Therefore there cannot be two ſimilar ſegments of a circle upon the ſame ſide of the ſame line, which do not coincide. Q. E. D.



[a] 10. 3.

[b] 11. Def. 3.

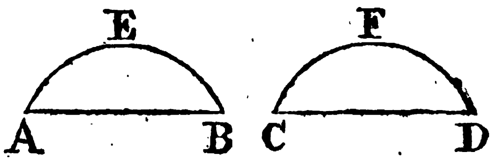
[c] 16. 1.

PROP. XXIV. THEOR.

SIMILAR ſegments of circles upon equal ſtraight lines, are equal to one another. See N.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; the ſegment AEB is equal to the ſegment CFD.

For if the ſegment AEB be applied to the ſegment CFD, ſo as the point A be on C, and the ſtraight line AB upon CD, the point B ſhall coincide with the point D,



becauſe AB is equal to CD. therefore the ſtraight line AB coinciding with CD, the ſegment AEB muſt [a] coincide with the ſegment CFD, and therefore is equal to it. Wherefore ſimilar ſegments, &c. Q. E. D.

a. 23. 3.

PROP. XXV. PROB.

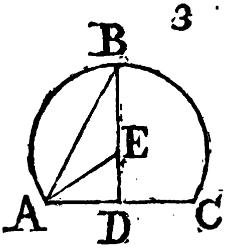
See N.

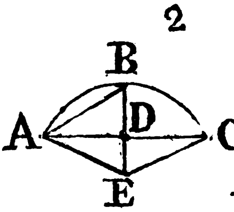
A Segment of a circle being given, to deſcribe the circle of which it is the ſegment.

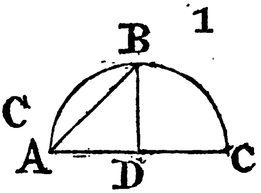
Let ABC be the given ſegment of a circle; it is required to deſcribe the circle of which it is the ſegment.

a. 10. 1.
b. 11. 1.
c. 6. 1.
d. 9. 3.

Biſect [a] AC in D, and from the point D draw [b] DB at right angles to AC, and join AB. Firſt, let the angles ABD, BAD be equal to one another; then the ſtraight line BD is equal [c] to DA, and therefore to DC. and becauſe the three ſtraight lines DA, DB, DC are all equal, D is the center of the circle [d]. from the center D, at the diſtance of any of the three DA, DB, DC deſcribe a circle; this ſhall paſs thro' the other points; and the circle of which ABC is a ſegment is deſcribed. and becauſe the center D is in AC,







the ſegment ABC is a ſemicircle. but if the angles ABD, BAD are not equal to one another, at the point A in the ſtraight line AB make [e] the angle BAE equal to the angle ABD, and produce BD to E, and join EC. and becauſe the angle ABE is equal to the angle BAE, the ſtraight line BE is equal [c] to EA. and becauſe AD is equal to DC, and DE common to the triangles ADE, CDE, the two ſides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the baſe AE is equal [f] to the baſe EC. but AE was ſhewn to be equal to EB, wherefore alſo BE is equal to EC; and the three ſtraight lines AE, EB, EC are therefore equal

e. 23. 1.
f. 4. 1.

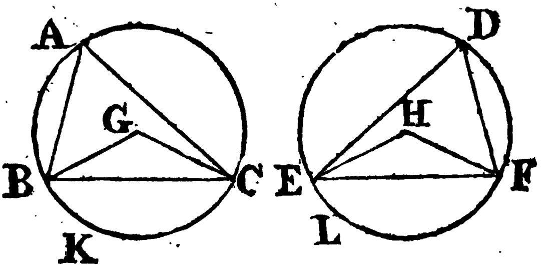
to one another; wherefore [d] E is the center of the circle. from the center E, at the diſtance of any of the three AE, EB, EC deſcribe a circle, this ſhall paſs thro' the other points; and the circle of which ABC is a ſegment is deſcribed. and it is evident that if the angle ABD be greater than the angle BAD, the center E falls without the ſegment ABC, which therefore is leſs than a ſemicircle. but if the angle ABD be leſs than BAD, the center E falls within the ſegment ABC, which is therefore greater than a ſemicircle. wherefore a ſegment of a circle being given, the circle is deſcribed of which it is a ſegment. Which was to be done. d. 9. 3.

PROP. XXVI. THEOR.

IN equal circles, equal angles ſtand upon equal circumferences, whether they be at the centers or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centers, and BAC, EDF at their circumferences. the circumference BKC is equal to the circumference ELF.

Join BC, EF; and becauſe the circles ABC, DEF are equal, the ſtraight lines drawn from their centers are equal; therefore the two ſides BG, GC, are equal to the two EH, HF; and the angle



at G is equal to the angle at H; therefore the baſe BC is equal [a] to the baſe EF. and becauſe the angle at A is equal to the angle at D, the ſegment BAC is ſimilar [b] to the ſegment EDF; and they are upon equal ſtraight lines BC, EF; but ſimilar ſegments of circles upon equal ſtraight lines are equal [c] to one another; therefore the ſegment BAC is equal to the ſegment EDF. but the whole circle ABC is equal to the whole DEF, therefore the

a. 4. 1.
b. 11. Def. 3.
c. 24. 3.

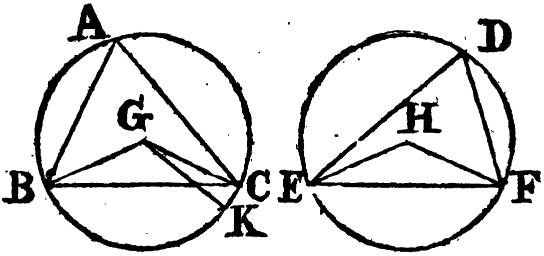
Book III. remaining ſegment BKC is equal to the remaining ſegment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVII. THEOR.

IN equal circles, the angles which ſtand upon equal circumferences, are equal to one another, whether they be at the centers, or circumferences.

Let the angles BGC, EHF at the centers, and BAC, EDF at the circumferences of the equal circles ABC, DEF ſtand upon the equal circumferences BC, EF. the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

a. 20. 3. If the angle BGC be equal to the angle EHF, it is manifeſt [a] that the angle BAC is alſo equal to EDF. but if not, one of them



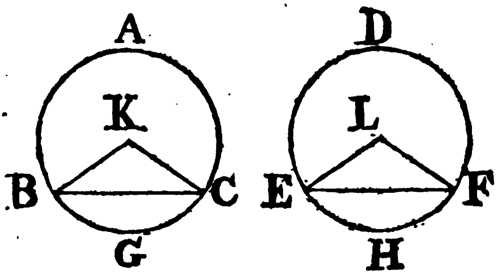
is the greater. let BGC be the greater, and at the point G, in the
b. 23. 1. ſtraight line BG, make [b] the angle BGK equal to the angle EHF;
c. 26. 3. but equal angles ſtand upon equal circumferences [c], when they are at the center; therefore the circumference BK is equal to the circumference EF. but EF is equal to BC, therefore alſo BK is equal to BC, the leſs to the greater, which is impoſſible. therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it. and the angle at A is half of the angle BGC, and the angle at D half of the angle EHF. therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVIII. THEOR.

IN equal circles, equal ſtraight lines cut off equal circumferences, the greater equal to the greater, and the leſs to the leſs.

Let ABC, DEF be equal circles, and BC, EF equal ſtraight lines in them, which cut off the two greater circumferences BAC, EDF, and the two leſs BGC, EHF. the greater BAC is equal to the greater EDF, and the leſs BGC to the leſs EHF.

Take [a] K, L the centers of the circles, and join BK, KC, EL, LF. and becauſe the circles are equal, the ſtraight lines from their centers are equal, therefore BK, KC, are equal to EL, LF; and the baſe BC is equal to the baſe EF; therefore the angle BKC is equal [b] to the angle ELF. but equal angles ſtand upon equal [c] circumferences, when they are at the centers; therefore the circumference BGC is equal to the circumference EHF. but the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D.



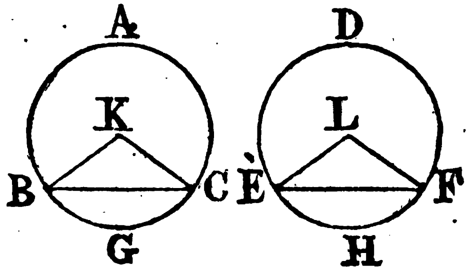
a. 1. 3.
b. 8. 1.
c. 26. 3.

PROP. XXIX. THEOR.

IN equal circles equal circumferences are ſubtended by equal ſtraight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF alſo be equal; and join BC, EF. the ſtraight line BC is equal to the ſtraight line EF.

a. 1. 3. Take [a] K, L the centers of the circles, and join BK, KC, EL, LF. and becauſe the circumference BGC is equal to the circum-



b. 27. 3. ference EHF, the angle BKC is equal [b] to the angle ELF. and becauſe the circles ABC, DEF are equal, the ſtraight lines from their centers are equal; therefore BK, KC are equal to EL, LF,
c. 4. 1. and they contain equal angles. therefore the baſe BC is equal [c] to the baſe EF. Therefore, in equal circles, &c. Q. E. D.

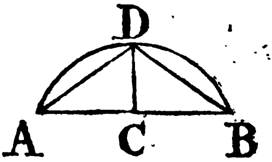
PROP. XXX. PROB.

TO biſect a given circumference, *that is, to divide it into two equal parts.*

Let ADB be the given circumference; it is required to biſect it.

a. 10. 1. Join AB, and biſect [a] it in C; from the point C draw CD at right angles to AB, and join AD, DB. the circumference ADB is biſected in the point D.

Becauſe AC is equal to CB, and CD common to the triangles ACD, BCD, the two ſides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becauſe each of them is a right angle; therefore the baſe AD is equal
b. 4. 1. [b] to the baſe BD. but equal ſtraight
c. 28. 3. lines cut off equal [c] circumferences, the greater equal to the greater, and the leſs to the leſs, and AD, DB are each of them leſs than a
d. Cor. 1. 3. ſemicircle; becauſe DC paſſes through the center [d]. wherefore the circumference AD is equal to the circumference DB. therefore the given circumference is biſected in D. Which was to be done.



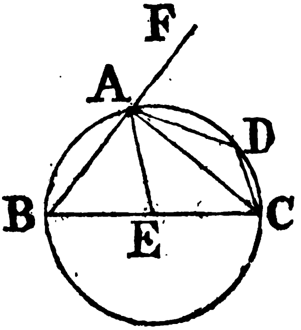
PROP. XXXI. THEOR.

IN a circle, the angle in a ſemicircle is a right angle; but the angle in a ſegment greater than a ſemicircle is leſs than a right angle; and the angle in a ſegment leſs than a ſemicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and center E; and draw CA dividing the circle into the ſegments ABC, ADC, and join BA, AD, DC. the angle in the ſemicircle BAC is a right angle; and the angle in the ſegment ABC, which is greater than a ſemicircle, is leſs than a right angle; and the angle in the ſegment ADC which is leſs than a ſemicircle is greater than a right angle.

Join AE, and produce BA to F; and becauſe BE is equal to EA, the angle EAB is equal [a] to EBA; alſo, becauſe AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB. but FAC, the exterior angle of the triangle ABC, is equal [b] to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right [c] angle. wherefore the angle BAC in a ſemicircle is a right angle.

a. 5. 1.
b. 32. 1.
c. 10. Def. 1.



And becauſe the two angles ABC, BAC of the triangle ABC are together leſs [d] than two right angles, and that BAC is a right angle, ABC muſt be leſs than a right angle; and therefore the angle in a ſegment ABC greater than a ſemicircle, is leſs than a right angle.

d. 17. 1.

And becauſe ABCD is a quadrilateral figure in a circle, any two of its oppoſite angles are equal [e] to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is leſs than a right angle, wherefore the other ADC is greater than a right angle.

e. 22. 3.

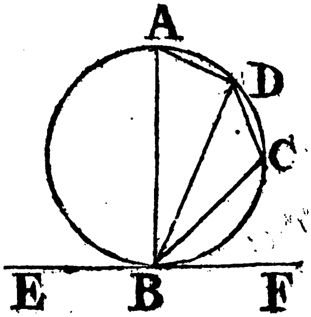
Beſides, it is manifeſt, that the circumference of the greater ſegment ABC falls without the right angle CAB, but the circumference of the leſs ſegment ADC falls within the right angle CAF. 'And this is all that is meant, when in the Greek text, and the 'tranſlations from it, the angle of the greater ſegment is ſaid to 'be greater, and the angle of the leſs ſegment is ſaid to be leſs 'than a right angle.'

Cor. From this it is manifeſt, that if one angle of a triangle be equal to the other two, it is a right angle, becauſe the angle adjacent to it is equal to the ſame two; and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

If a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn cutting the circle, the angles made by this line with the line touching the circle, ſhall be equal to the angles which are in the alternate ſegments of the circle.

Let the ſtraight line EF touch the circle ABCD in B, and from the point B let the ſtraight line BD be drawn cutting the circle, the angles which BD makes with the touching line EF ſhall be equal to the angles in the alternate ſegments of the circle; that is, the angle FBD is equal to the angle which is in the ſegment DAB, and the angle DBE to the angle in the ſegment BCD.



a. 11. 1. From the point B draw [a] BA at right angles to EF, and take any point C in the circumference BD, and join AD, DC, CB; and becauſe the ſtraight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the center of the circle b. 19. 3. is [b] in BA; therefore the angle ADB c. 31. 3. in a ſemicircle is a right [c] angle, and conſequently the other two angles d. 32. 1. BAD, ABD are equal [d] to a right angle. but ABF is likewiſe a right angle; therefore the angle ABF is equal to the angles BAD, ABD. take from theſe equals the common angle ABD; therefore the remaining angle DBF is equal to the angle

BAD, which is in the alternate ſegment of the circle; and becauſe ABCD is a quadrilateral figure in a circle, the oppoſite angles BAD, BCD are equal [e] to two right angles; therefore the angles DBF, DBE, being likewiſe equal [f] to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD in the alternate ſegment of the circle. Wherefore, if a ſtraight line, &c. Q. E. D.

e. 22. 3.
f. 13. 1.

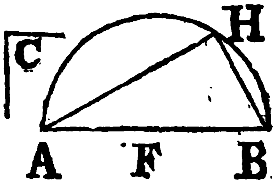
PROP. XXXIII. PROB.

UPON a given ſtraight line to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle.

See N.

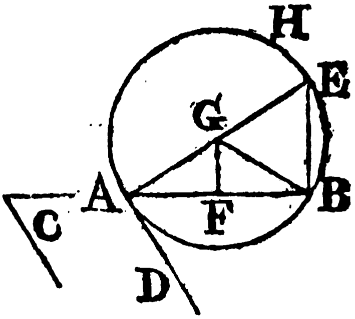
Let AB be the given ſtraight line, and the angle at C the given rectilineal angle; it is required to deſcribe upon the given ſtraight line AB a ſegment of a circle, containing an angle equal to the angle C.

Firſt, Let the angle at C be a right angle, and biſect [a] AB in F, and from the center F, at the diſtance FB, deſcribe the ſemicircle AHB; therefore the angle AHB in a ſemicircle is [b] equal to the right angle at C.



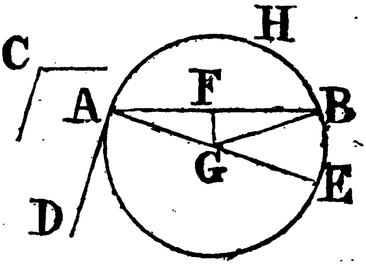
a. 10. 1.
b. 31. 3.

But if the angle C be not a right angle, at the point A in the ſtraight line AB, make [c] the angle BAD equal to the angle C, and from the point A draw [d] AE at right angles to AD; biſect [a] AB in F, and from F draw [d] FG at right angles to AB, and join GB. and becauſe AF is equal to FB, and FG common to the triangles AFG, BFG, the two ſides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the baſe AG is equal [e] to the baſe GB; and the circle deſcribed from the center G, at



c. 23. 1.
d. 11. 1.
e. 4. 1.

Book III. the diſtance GA, ſhall paſs through the point B; let this be the
circle AHB. and becauſe from the point A the extremity of the
f. Cor. 16. 3. diameter AE, AD is drawn at right angles to AE, therefore AD [f]
touches the circle; and becauſe
AB drawn from the point of
contact A cuts the circle, the
angle DAB is equal to the angle in the alternate ſegment
g. 32. 3. AHB [g]. but the angle DAB is
equal to the angle C, therefore
alſo the angle C is equal to the
angle in the ſegment AHB.
wherefore upon the given ſtraight line AB the ſegment AHB of
a circle is deſcribed which contains an angle equal to the given
angle at C. Which was to be done.

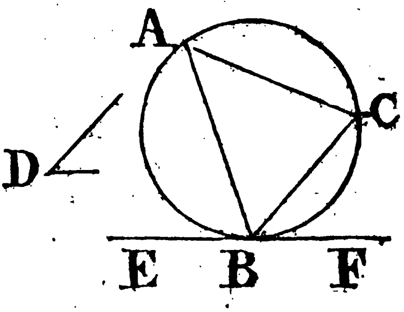


PROP. XXXIV. PROB.

TO cut off a ſegment from a given circle which ſhall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a ſegment from the circle ABC that ſhall contain an angle equal to the angle D.

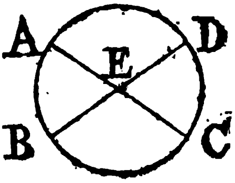
a. 17. 3. Draw [a] the ſtraight line EF touching the circle ABC in the
b. 23. 1. point B, and at the point B, in the ſtraight line BF, make [b] the
angle FBC equal to the angle D. therefore, becauſe the
ſtraight line EF touches the
circle ABC, and BC is drawn
from the point of contact B,
c. 32. 3. the angle FBC is equal [c] to
the angle in the alternate
ſegment BAC of the circle.
but the angle FBC is equal
to the angle D; therefore
the angle in the ſegment BAC is equal to the angle D. wherefore
the ſegment BAC is cut off from the given circle ABC containing an angle equal to the given angle D. Which was to be
done.



PROP. XXXV. THEOR.

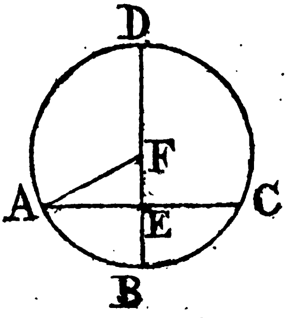
IF two ſtraight lines within a circle cut one another, the rectangle contained by the ſegments of one of them is equal to the rectangle contained by the ſegments of the other. See N.

Let the two ſtraight lines AC, BD, within the circle ABCD, cut one another in the point E; the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.



If AC, BD paſs each of them through the center, ſo that E is the center; it is evident, that AE, EC, BE, ED, being all equal, the rectangle AE, EC is likewiſe equal to the rectangle BE, ED.

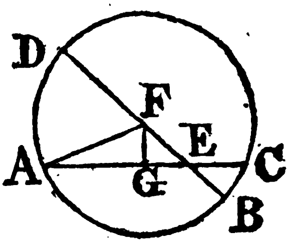
But let one of them BD paſs thro' the center, and cut the other AC, which does not paſs thro' the center, at right angles, in the point E. then, if BD be biſected in F, F is the center of the circle ABCD; and join AF. and becauſe BD, which paſſes thro' the center, cuts the ſtraight line AC, which does not paſs thro' the center, at right angles in E, AE, EC are equal [a] to one another. a. 3. 3. and becauſe the ſtraight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED together with the ſquare of EF, is equal [b] to the ſquare of FB; b. 5. 2. that is, to the ſquare of FA; but the ſquares of AE, EF are equal [c] to the ſquare of FA; c. 47. 1. therefore the rectangle BE, ED, together with the ſquare of EF, is equal to the ſquares of AE, EF. take away the common ſquare of EF, and the remaining rectangle BE, ED is equal to the remaining ſquare of AE; that is, to the rectangle AE, EC.



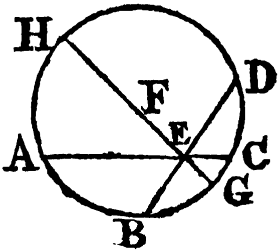
Next, Let BD which paſſes thro' the center, cut the other AC, which does not paſs thro' the center, in E, but not at right angles. then, as before, if BD be biſected in F, F is the center of the circle. Join AF, and from F draw [d] FG perpendicular to AC; therefore d. 12. 1.

a. 3. 3.
b. 5. 2.
c. 47. 1.

AG is equal [a] to GC; wherefore the rectangle AE, EC, together with the ſquare of EG, is equal [b] to the ſquare of AG. to each of theſe equals add the ſquare of GF, therefore the rectangle AE, EC, together with the ſquares of EG, GF is equal to the ſquares of AG, GF. but the ſquares of EG, GF are equal [c] to the ſquare of EF; and the ſquares of AG, GF are equal to the ſquare of AF. therefore the rectangle AE, EC, together with the ſquare of EF is equal to the ſquare of AF; that is, to the ſquare of FB. but the ſquare of FB is equal [b] to the rectangle BE, ED together with the ſquare of EF; therefore the rectangle AE, EC, together with the ſquare of EF, is equal to the rectangle BE, ED together with the ſquare of EF. take away the common ſquare of EF, and the remaining rectangle AE, EC is therefore equal to the remaining rectangle BE, ED.



Laſtly, Let neither of the ſtraight lines AC, BD paſs thro' the center. take the center F, and thro' E the interſection of the ſtraight lines AC, DB draw the diameter GEFH. and becauſe the rectangle AE, EC is equal, as has been ſhewn, to the rectangle GE, EH; and for the ſame reaſon, the rectangle BE, ED is equal to the ſame rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two ſtraight lines, &c. Q. E. D.



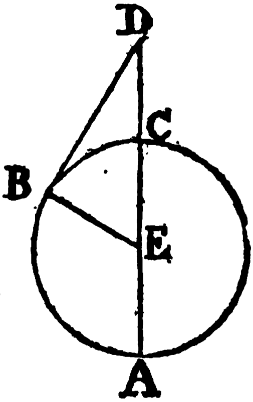
PROP. XXXVI. THEOR.

IF from any point without a circle two ſtraight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, ſhall be equal to the ſquare of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two ſtraight lines drawn from it, of which DCA cuts the circle, and DB touches the ſame. the rectangle AD, DC is equal to the ſquare of DB.

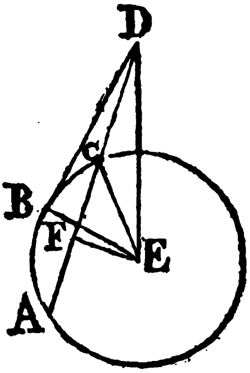
Either DCA paſſes thro' the center, or it does not; firſt, let it paſs thro' the center E, and join EB; therefore the angle EBD is a right [a] angle. and becauſe the ſtraight line AC is biſected in E, and produced to the point D, the rectangle AD, DC together with the ſquare of EC, is equal [b] to the ſquare of ED. and CE is equal to EB, therefore the rectangle AD, DC, together with the ſquare of EB is equal to the ſquare of ED. but the ſquare of ED is equal [c] to the ſquares of EB, BD, becauſe EBD is a right angle. therefore the rectangle AD, DC together with the ſquare of EB, is equal to the ſquares of EB, BD. take away the common ſquare of EB; therefore the remaining rectangle AD, DC is equal to the ſquare of the tangent DB.

a. 18. 3.
b. 6. 2.
c. 47. 1.

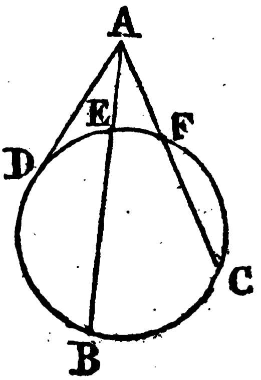


But if DCA does not paſs thro' the center of the circle ABC, take [d] the center E, and draw EF perpendicular [e] to AC, and join EB, EC, ED; then EFD is a right angle. and becauſe the ſtraight line EF, which paſſes thro' the center, cuts the ſtraight line AC, which does not paſs thro' the center, at right angles, it ſhall likewiſe biſect [f] it; therefore AF is equal to FC. and becauſe the ſtraight line AC is biſected in F, and produced to D, the rectangle AD, DC together with the ſquare of FC, is equal [b] to the ſquare of FD. to each of theſe equals add the ſquare of FE, therefore the rectangle AD, DC together with the ſquares of CF, FE, is equal to the ſquares of DF, FE. but the ſquare of ED is equal [c] to the ſquares of DF, FE, becauſe EFD is a right angle; and the ſquare of EC is equal to the ſquares of CF, FE; therefore the rectangle AD, DC together with the ſquare of EC, is equal to the ſquare of ED. and CE is equal to EB, therefore the rectangle AD, DC toge-

d. 1. 3.
e. 12. 1.
f. 3. 3.



Book III. ther with the ſquare of EB, is equal to the ſquare of ED. but
the ſquares of EB, BD are equal to the ſquare [c] of ED, becauſe
c. 47. 1. EBD is a right angle; therefore the rectangle AD, DC together with the ſquare of EB, is equal to the ſquares of EB, BD. take away the common ſquare of EB, therefore the remaining rectangle AD, DC is equal to the ſquare of DB. Wherefore, if from any point, &c. Q. E. D.



Cor. If from any point without a circle, there be drawn two ſtraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF. for each of them is equal to the ſquare of the ſtraight line AD which touches the circle.

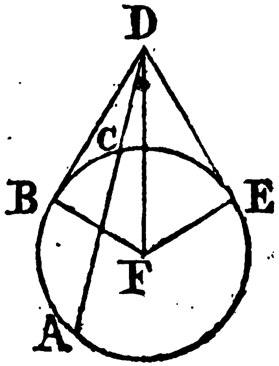
PROP. XXXVII. THEOR.

If from a point without a circle there be drawn two ſtraight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the ſquare of the line which meets it, the line which meets ſhall touch the circle.

Let any point D be taken without the circle ABC, and from it let two ſtraight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC be equal to the ſquare of DB; DB touches the circle.

a. 17. 3. Draw [a] the ſtraight line DE touching the circle ABC, find its
b. 18. 3. center F, and join FE, FB, FD; then FED is a right [b] angle. and
becauſe DE touches the circle ABC, and DCA cuts it, the rec-
c. 36. 3. tangle AD, DC is equal [c] to the ſquare of DE. but the rectangle
AD, DC is, by hypotheſis, equal to the ſquare of DB; therefore the ſquare of DE is equal to the ſquare of DB, and the ſtraight line DE equal to the ſtraight line DB. and FE is equal to FB,

wherefore DE, EF are equal to DB, BF; and the baſe FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal [d] to the angle DBF, but DEF is a right angle, therefore alſo DBF is a right angle, and FB, if produced, is a diameter, and the ſtraight line which is drawn at right angles to a diameter, from the extremity of it, touches [e] the circle. therefore DB touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.



d. 8. 1.

e. 16. 3.

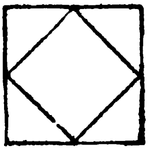
THE ELEMENTS OF EUCLID.

BOOK IV.

DEFINITIONS.

I.

A Rectilineal figure is ſaid to be inſcribed in another rectilineal figure, when all the angles of the inſcribed figure are upon the ſides of the figure in which it is inſcribed, each upon each.

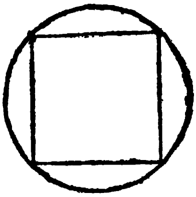


II.

In like manner, a figure is ſaid to be deſcribed about another figure, when all the ſides of the circumſcribed figure paſs thro' the angular points of the figure about which it is deſcribed, each thro' each.

III.

A rectilineal figure is ſaid to be inſcribed in a circle, when all the angles of the inſcribed figure are upon the circumference of the circle.

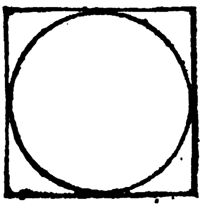


IV.

A rectilineal figure is ſaid to be deſcribed about a circle, when each ſide of the circumſcribed figure touches the circumference of the circle.

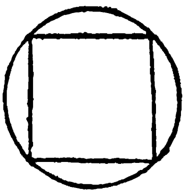
V.

In like manner a circle is ſaid to be inſcribed in a rectilineal figure, when the circumference of the circle touches each ſide of the figure.



VI.

A circle is ſaid to be deſcribed about a rectilineal figure, when the circumference of the circle paſſes thro' all the angular points of the figure about which it is deſcribed.



VII.

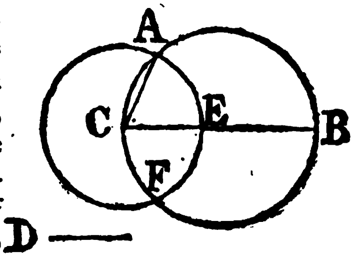
A ſtraight line is ſaid to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a ſtraight line, equal to a given ſtraight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given ſtraight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a ſtraight line BC is placed equal to D. but if it is not, BC is greater than D; make CE equal [a] to D, and from the center C, at the diſtance CE, deſcribe the circle AEF, and join CA. therefore, becauſe C is the center of the circle AEF, CA is equal to CE; but D is equal to CE, therefore D is equal to CA. wherefore in the circle ABC a ſtraight line is placed equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done.



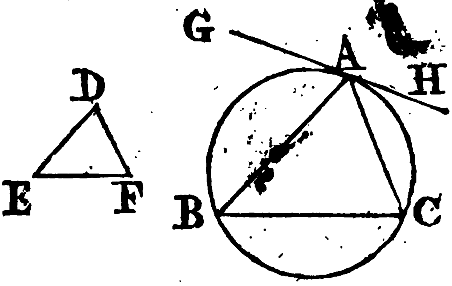
a. 3. 1.

PROP. II. PROB.

IN a given circle to inſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe in the circle ABC a triangle equiangular to the triangle DEF.

a. 17. 3. Draw [a] the ſtraight line GAH touching the circle in the point
b. 23. 1. A, and at the point A, in the ſtraight line AH, make [b] the angle HAC equal to the angle DEF; and at the point A, in the ſtraight line AG, make the angle GAB equal to the angle DFE, and join BC. therefore becauſe HAG touches the circle ABC, and AC is drawn from the point of contact, the angle
c. 32. 3. HAC is equal [c] to the angle ABC in the alternate ſegment of the circle. but HAC is equal to the angle DEF, therefore alſo the angle ABC is equal to DEF. for the ſame reaſon, the angle ACB is equal to the angle DFE; therefore the
d. 32. 1. remaining angle BAC is equal [d] to the remaining angle EDF. wherefore the triangle ABC is equiangular to the triangle DEF, and it is inſcribed in the circle ABC. Which was to be done.



PROP. III. PROB.

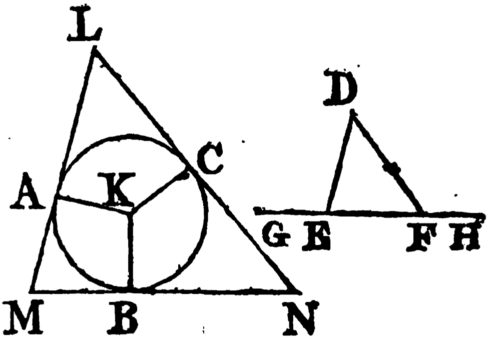
ABOUT a given circle to deſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to deſcribe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points G, H, and find the center K of the circle ABC, and from it draw any ſtraight line KB; at
a. 23. 1. the point K in the ſtraight line KB, make [a] the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and thro' the points A, B, C draw the ſtraight lines LAM,
b. 17. 3. MBN, NCL touching [b] the circle ABC. therefore becauſe LM, MN, NL touch the circle ABC in the points A, B, C to which from the center are drawn KA, KB, KC, the angles at the points
c. 18. 3. A, B, C are right [c] angles. and becauſe the four angles of the

quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles; and that two of them KAM, KBM are right angles, the other two AKB, AMB are equal to two right angles. but the angles DEG, DEF are likewiſe equal [d] to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF. in like manner the angle LNM may be demonſtrated to be equal to DFE; and therefore the remaining angle MLN is equal [e] to the remaining angle EDF. wherefore the triangle LMN is equiangular to the triangle DEF. and it is deſcribed about the circle ABC. Which was to be done.

d. 13. 1.
e. 32. 1.



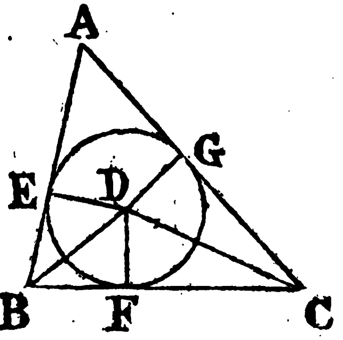
PROP. IV. PROB.

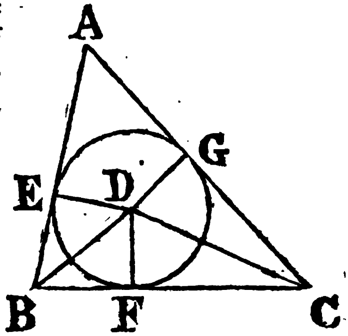
To inſcribe a circle in a given triangle. See N.

Let the given triangle be ABC; it is required to inſcribe a circle in ABC.

Biſect [a] the angles ABC, BCA by the ſtraight lines BD, CD meeting one another in the point D, from which draw [b] DE, DF, DG perpendiculars to AB, BC, CA. and becauſe the angle EBD is equal to the angle FBD, for the angle ABC is biſected by BD, and that the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the ſide BD, which is oppoſite to one of the equal angles in each, is common to both; therefore their other ſides ſhall be equal [c]; wherefore

a. 9. 1.
b. 12. 1.
c. 26. 1.





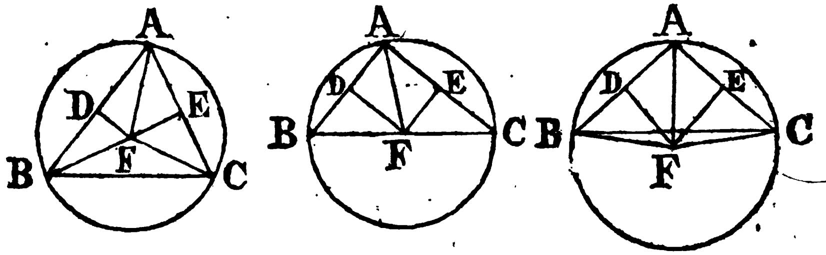
DE is equal to DF. for the ſame reaſon, DG is equal to DF; therefore the three ſtraight lines DE, DF, DG are equal to one another, and the circle deſcribed from the center D, at the diſtance of any of them, ſhall paſs thro' the extremities of the other two, and touch the ſtraight lines AB, BC, CA, becauſe the angles at the points E, F, G are right angles, and the ſtraight line which is drawn from the extremity of a diameter at right d. 16. 3. angles to it, touches [d] the circle. therefore the ſtraight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inſcribed in the triangle ABC. Which was to be done.

PROB. V. PROB.

See N. TO deſcribe a circle about a given triangle.

Let the given triangle be ABC; it is required to deſcribe a circle about ABC.

a. 10. 1. Biſect [a] AB, AC in the points D, E, and from theſe points b. 11. 1. draw DF, EF at right angles [b] to AB, AC; DF, EF produced meet one another. for if they do not meet they are parallel, wherefore AB, AC which are at right angles to them are parallel; which is abſurd. let them meet in F, and join FA; alſo, if the point F be not in BC, join BF, CF. then becauſe AD is equal to DB, and DF common, and at right angles to AB, the baſe AF is c. 4. 1. equal [c] to the baſe FB. in like manner it may be ſhewn that CF is equal to FA; and therefore BF is equal to FC; and FA, FB,



FC are therefore equal to one another. wherefore the circle deſcribed from the center F, at the diſtance of one of them, ſhall paſs thro' the extremities of the other two; and be deſcribed about the triangle ABC. Which was to be done.

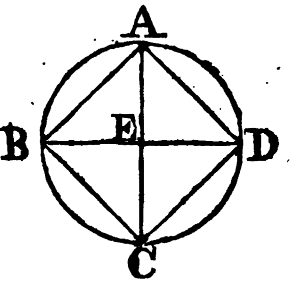
Cor. And it is manifeſt that when the center of the circle falls within the triangle, each of its angles is leſs than a right angle, each of them being in a ſegment greater than a ſemicircle. but when the center is in one of the ſides of the triangle, the angle oppoſite to this ſide, being in a ſemicircle, is a right angle. and if the center falls without the triangle, the angle oppoſite to the ſide beyond which it is, being in a ſegment leſs than a ſemicircle, is greater than a right angle. Wherefore, if the given triangle be acute angled, the center of the circle falls within it; if it be a right angled triangle, the center is in the ſide oppoſite to the right angle; and if it be an obtuſe angled triangle, the center falls without the triangle, beyond the ſide oppoſite to the obtuſe angle.

PROP. VI. PROB.

TO inſcribe a ſquare in a given circle.

Let ABCD be the given circle; it is required to inſcribe a ſquare in ABCD.

Draw the diameters AC, BD at right angles to one another; and join AB, BC, CD, DA. becauſe BE is equal to ED, for E is the center, and that EA is common, and at right angles to BD; the baſe BA is equal [a] to the baſe AD. and for the ſame reaſon, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is alſo rectangular; for the ſtraight line BD being the diameter of the circle ABCD, BAD is a ſemicircle; wherefore the angle BAD is a right [b] angle. for the ſame reaſon each of the angles ABC, BCD, CDA is a right angle. therefore the quadrilateral figure ABCD is rectangular. and it has been ſhewn to be equilateral, therefore it is a ſquare; and it is inſcribed in the circle ABCD. Which was to be done.



a. 4. 1.

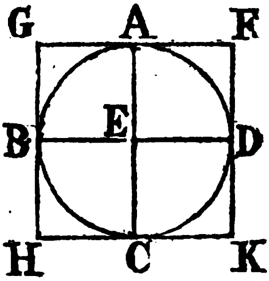
b. 31. 3.

PROP. VII. PROB.

TO deſcribe a ſquare about a given circle.

Let ABCD be the given circle; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at right
a. 17. 3. angles to one another, and thro' the points A, B, C, D draw [a] FG,
GH, HK, KF touching the circle. and becauſe FG touches the
circle ABCD, and EA is drawn from the center E to the point of
b. 18. 3. contact A, the angles at A are right [b] angles. for the ſame reaſon,
the angles at the points B, C, D are right angles. and becauſe the
angle AEB is a right angle, as likewiſe
c. 28. 1. is EBG, GH is parallel [c] to AC. for the
ſame reaſon, AC is parallel to FK. and
in like manner GF, HK may each of them
be demonſtrated to be parallel to BED.
therefore the figures GK, GC, AK, FB,
BK are parallelograms, and GF is there-
d. 34. 1. fore equal [d] to HK, and GH to FK.
and becauſe AC is equal to BD, and
that AC is equal to each of the two GH, FK; and BD to each of
the two GF, HK; GH, FK are each of them equal to GF or HK.
therefore the quadrilateral figure FGHK is equilateral. It is alſo
rectangular; for GBEA being a parallelogram, and AEB a right
angle, AGB is [d] likewiſe a right angle. in the ſame manner it may
be ſhewn that the angles at H, K, F are right angles. therefore the
quadrilateral figure FGHK is rectangular. and it was demonſtrated
to be equilateral; therefore it is a ſquare; and it is deſcribed about
the circle ABCD. Which was to be done.



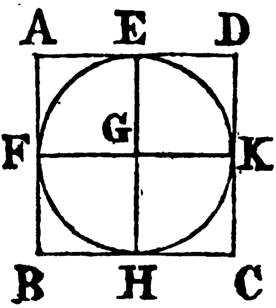
PROP. VIII. PROB.

TO inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare; it is required to inſcribe a circle in ABCD.

a. 10. 1. Biſect [a] each of the ſides AB, AD, in the points F, E, and thro' E
b. 31. 1. draw [b] EH parallel to AB or DC, and thro' F draw FK parallel to

AD or BC. therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their oppoſite ſides are equal[c]. and becauſe AD is equal to AB, and that AE is the half of AD, c. 34. 1.
and AF the half of AB; AE is equal to AF. wherefore the ſides oppoſite to theſe are equal, viz. FG to GE. in the ſame manner it may be demonſtrated that GH, GK are each of them equal to FG or GE. therefore the four ſtraight lines GE, GF, GH, GK are equal to one another; and the circle deſcribed from the center G, at the diſtance of one of them ſhall paſs thro' the extremities of the other three, and touch the ſtraight lines AB, BC, CD, DA; becauſe the angles at the points E, F, H, K are right[d] angles, and that the d. 29. 1.
ſtraight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle[e]. therefore each of the e. 16. 3.
ſtraight lines AB, BC, CD, DA touches the circle, which therefore is inſcribed in the ſquare ABCD. Which was to be done.

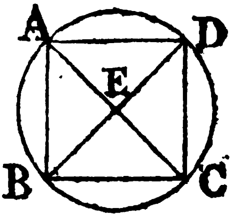


PROP. IX. PROB.

TO deſcribe a circle about a given ſquare.

Let ABCD be the given ſquare; it is required to deſcribe a circle about it.

Join AC, BD cutting one another in E. and becauſe DA is equal to AB, and AC common to the triangles DAC, BAC, the two ſides DA, AC are equal to the two BA, AC; and the baſe DC is equal to the baſe BC; wherefore the angle DAC is equal[a] to the a. 8. 1.
angle BAC, and the angle DAB is biſected by the ſtraight line AC. in the ſame manner it may be demonſtrated that the angles ABC, BCD, CDA are ſeverally biſected by the ſtraight lines BD, AC. therefore becauſe the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the ſide EA is equal[b] to the b. 6. 1.
ſide EB. in the ſame manner it may be demonſtrated that the ſtraight



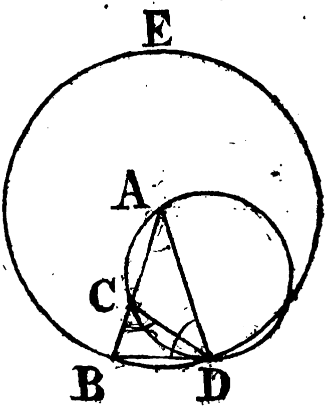
lines EC, ED are each of them equal to EA or EB. therefore the four ſtraight lines EA, EB, EC, ED are equal to one another; and the circle deſcribed from the center E, at the diſtance of one of them, ſhall paſs thro' the extremities of the other three, and be deſcribed about the ſquare ABCD. Which was to be done.

PROP. X. PROB.

TO deſcribe an iſoſceles triangle, having each of the angles at the baſe double of the third angle.

a. 11. 2. Take any ſtraight line AB, and divide [a] it in the point C, ſo
that the rectangle AB, BC be equal to the ſquare of CA; and
from the center A, at the diſtance AB deſcribe the circle BDE,
b. 1. 4. in which place [b] the ſtraight line BD equal to AC, which is not
greater than the diameter of the circle BDE; join DA, DC, and
c. 5. 4. about the triangle ADC deſcribe [c] the circle ACD. the triangle
ABD is ſuch as is required, that is, each of the angles ABD,
ADB is double of the angle BAD.

Becauſe the rectangle AB, BC is equal to the ſquare of AC, and
that AC is equal to BD, the rectangle AB, BC is equal to the
ſquare of BD. and becauſe from
the point B without the circle
ACD two ſtraight lines BCA, BD
are drawn to the circumference,
one of which cuts, and the other
meets the circle, and that the rec-
tangle AB, BC contained by the
whole of the cutting line, and the
part of it without the circle, is
equal to the ſquare of BD which
meets it; the ſtraight line BD
d. 37. 3. touches [d] the circle ACD. and
becauſe BD touches the circle,
and DC is drawn from the point
e. 32. 3. of contact D, the angle BDC is equal [e] to the angle DAC in the
alternate ſegment of the circle; to each of theſe add the angle
CDA, therefore the whole angle BDA is equal to the two angles
f. 32. 1. CDA, DAC. but the exterior angle BCD is equal [f] to the angles
CDA, DAC; therefore alſo BDA is equal to BCD, but BDA is



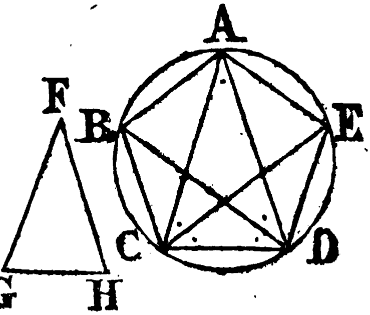
equal [g] to the angle CBD, becauſe the ſide AD is equal to the ſide AB; therefore CBD, or DBA is equal to BCD; and conſequently
the three angles BDA, DBA, BCD are equal to one another. g. 5. 1.
and becauſe the angle DBC is equal to the angle BCD, the
ſide BD is equal [h] to the ſide DC. but BD was made equal to CA, h. 6. 1.
therefore alſo CA is equal to CD, and the angle CDA equal [g] to the angle DAC. therefore the angles CDA, DAC together, are double of the angle DAC. but BCD is equal to the angles CDA, DAC; therefore alſo BCD is double of DAC. and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB. wherefore an iſoſceles triangle ABD is deſcribed having each of the angles at the baſe double of the third angle. Which was to be done.

PROP. XI. PROB.

TO inſcribe an equilateral and equiangular pentagon in a given circle.

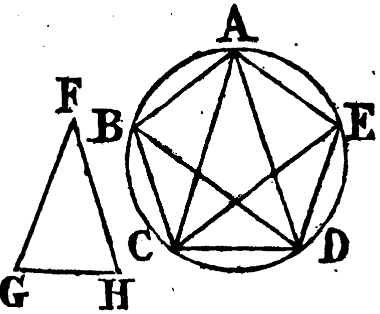
Let ABCDE be the given circle; it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

Deſcribe [a] an Iſoſceles triangle FGH having each of the angles a. 10. 4.
at G, H double of the angle at F; and in the circle ABCDE inſcribe [b] the triangle ACD equiangular to the triangle FGH, ſo b. 2. 4.
that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. Biſect [c] the angles ACD, CDA by the ſtraight lines CE, DB, and join AB, BC, DE, EA. ABCDE is the pentagon required. c. 9. 1.



Becauſe each of the angles ACD, CDA is double of CAD, and are biſected by the ſtraight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another. but equal angles
ſtand upon equal [d] circumferences; therefore the five circumferences d. 26. 3.
AB, BC, CD, DE, EA are equal to one another. and equal

circumferences are ſubtended by equal [e] ſtraight lines; therefore the five ſtraight lines AB, BC, CD, DE, EA are equal to one e. 29. 3. another. Wherefore the pentagon ABCDE is equilateral. It is alſo equiangular; becauſe the circumference AB is equal to the circumference DE, if to each be added BCD, the whole ABCD is equal to the whole EDCB. and the angle AED ſtands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefore the angle BAE is e- f. 27. 3. qual [f] to the angle AED. for the ſame reaſon, each of the angles ABC, BCD, CDE is equal to the angle BAE or AED. therefore the pentagon ABCDE is equiangular; and it has been ſhewn that it is equilateral. Wherefore in the given circle an equilateral and equiangular pentagon has been inſcribed. Which was to be done.

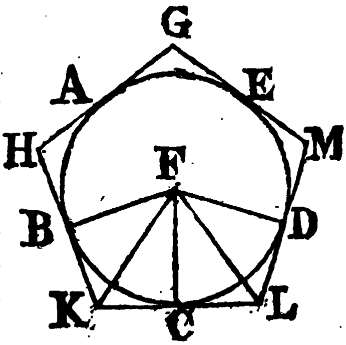


PROP. XII. PROB.

TO deſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to deſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon inſcribed in the circle, by the laſt propoſition, be in the points A, B, C, D, E, ſo that the circumfe- a. 11. 4. rences AB, BC, CD, DE, EA are equal [a]; and thro' the points A, b. 17. 3. B, C, D, E draw GH, HK, KL, LM, MG touching [b] the circle; take the center F, and join FB, FK, FC, FL, FD. and becauſe the ſtraight line KL touches the circle ABCDE in the point C, to c. 18. 3. which FC is drawn from the center F, FC is perpendicular [c] to KL; therefore each of the angles at C is a right angle. for the ſame reaſon, the angles at the points B, D are right angles. and becauſe FCK d. 47. 1. is a right angle, the ſquare of FK is equal [d] to the ſquares of FC, CK. for the ſame reaſon the ſquare of FK is equal to the ſquares of FB, BK. therefore the ſquares of FC, CK are equal to the ſquares of FB, BK, of which the ſquare of FC is equal to the ſquare of FB; the remaining ſquare of CK is therefore equal to the remaining ſquare of BK, and the ſtraight line CK equal to BK. and becauſe FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK; and the baſe BK is equal to the baſe KC; therefore the angle BFK is equal [e] to the angle e. 8. 1. KFC, and the angle BKF to FKC. wherefore the angle BFC is double of the angle KFC, and BKC double of FKC. for the ſame reaſon, the angle CFD is double of the angle CFL, and CLD double of CLF. and becauſe the circumference BC is equal to the circumference CD, the angle BFC is equal [f] to the angle CFD. f. 27. 3. and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL. therefore in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each, and the ſide FC, which is adjacent to the equal angles in each, is common to both; therefore the other ſides ſhall be equal [g] to the other ſides, g. 26. 1. and the third angle to the third angle. therefore the ſtraight line KC is equal to CL, and the angle FKC to the angle FLC. and becauſe KC is equal to CL, KL is double of KC. in the ſame manner, it may be ſhewn that HK is double of BK. and becauſe BK is equal to KC, as was demonſtrated, and that KL is double of KC, and HK double of BK, HK ſhall be equal to KL. in like manner it may be ſhewn that GH, GM, ML are each of them equal to HK or KL. therefore the pentagon GHKLM is equilateral. It is alſo equiangular; for ſince the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonſtrated; the angle HKL is equal to KLM. and in like manner it may be ſhewn, that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM. therefore the five angles GHK, HKL, KLM, LMG, MGH being equal to one another, the pentagon GHKLM is equiangular. and it is equilateral, as was demonſtrated; and it is deſcribed about the circle ABCDE. Which was to be done.

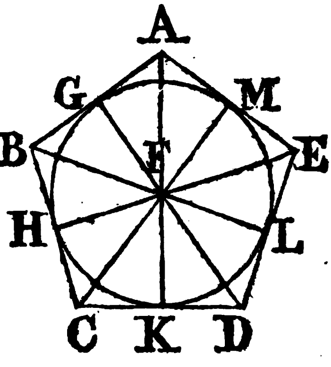


PROP. XIII. PROB.

TO inſcribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inſcribe a circle in the pentagon ABCDE.

a. 9. 1. Biſect [a] the angles BCD, CDE by the ſtraight lines CF, DF, and from the point F in which they meet draw the ſtraight lines FB, FA, FE. therefore ſince BC is equal to CD, and CF common to the triangles BCF, DCF, the two ſides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle DCF; therefore b. 4. 1. the baſe BF is equal [b] to the baſe FD, and the other angles to the other angles, to which the equal ſides are oppoſite; therefore the angle CBF is equal to the angle CDF. and becauſe the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is alſo double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is biſected by the ſtraight line BF. in the ſame manner it may be demonſtrated that the angles BAE, AED are biſected by the ſtraight lines AF, FE. c. 12. 1. from the point F draw [c] FG, FH, FK, FL, FM perpendiculars to the ſtraight lines AB, BC, CD, DE, EA. and becauſe the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC, FKC there are two angles of one equal to two angles of the other; and the ſide FC, which is oppoſite to one of the equal angles in each, is common to both; d. 26. 1. therefore the other ſides ſhall be equal [d], each to each; wherefore the perpendicular FH is equal to the perpendicular FK. in the ſame manner it may be demonſtrated that FL, FM, FG are each of them equal to FH or FK; therefore the five ſtraight lines FG, FH, FK, FL, FM are equal to one another. wherefore the circle deſcribed from the center F, at the diſtance of one of theſe five, ſhall paſs thro' the extremities of the other four, and touch the ſtraight



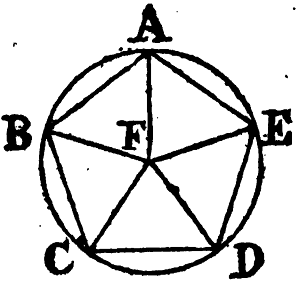
lines AB, BC, CD, DE, EA, becauſe the angles at the points G, H, K, L, M are right angles; and that a ſtraight line drawn from the extremity of the diameter of a circle at right angles to it, touches [c] the circle. therefore each of the ſtraight lines AB, BC, c. 16. 3. CD, DE, EA touches the circle; wherefore it is inſcribed in the pentagon ABCDE. Which was to be done.

PROP. XIV. PROB.

TO deſcribe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to deſcribe a circle about it.

Biſect [a] the angles BCD, CDE by the ſtraight lines CF, FD, a. 9. 1. and from the point F in which they meet draw the ſtraight lines FB, FA, FE to the points B, A, E. It may be demonſtrated, in the ſame manner as in the preceding propoſition, that the angles CBA, BAE, AED are biſected by the ſtraight lines FB, FA, FE. and becauſe the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the ſide CF is equal [b] to the ſide FD. in like manner it may be demon- b. 6. 1. ſtrated that FB, FA, FE are each of them equal to FC or FD. therefore the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; and the circle deſcribed from the center F, at the diſtance of one of them, ſhall paſs thro' the extremities of the other four, and be deſcribed about the equilateral and equiangular pentagon ABCDE. Which was to be done.



PROP. XV. PROB.

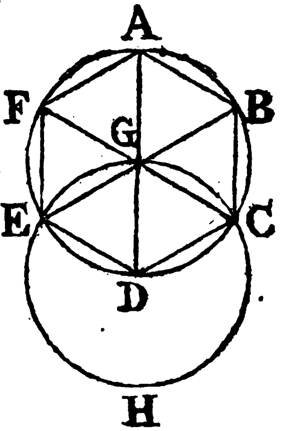
See N. TO inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in it.

Find the center G of the circle ABCDEF, and draw the diameter AGD; and from D as a center, at the diſtance DG deſcribe the circle EGCH, join EG, CG and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA. the hexagon ABCDEF is equilateral and equiangular.

Becauſe G is the center of the circle ABCDEF, GE is equal to GD. and becauſe D is the center of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral, and therefore its three angles EGD, GDE, DEG are equal to one another, becauſe the angles at the baſe of an iſoſceles triangle are equal [a]. and the three angles of a triangle are equal [b] to two right angles; therefore the angle EGD is the third part of two right angles. in the ſame manner it may be demonſtrated that the angle DGC is alſo the third part of two right angles. and becauſe the ſtraight line GC makes with EB the adjacent angles EGC, CGB equal [c] to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB are equal to one another. and to theſe are equal [d] the vertical oppoſite angles BGA, AGF, FGE. therefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another. but equal angles ſtand upon equal [e] circumferences; therefore the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another. and equal circumferences are ſubtended by equal [f] ſtraight lines; therefore the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is alſo equiangular; for ſince the circumference AF is equal to ED, to each of theſe add the circumference ABCD; therefore the whole circumference FABCD ſhall be equal to the whole EDCBA. and the angle FED ſtands upon

a. 5. 1.
b. 32. 1.
c. 13. 1.
d. 15. 1.
e. 26. 3.
f. 29. 3.



the circumference FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED. in the ſame manner it may be demonſtrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED. therefore the hexagon is equiangular. and it is equilateral, as was ſhewn; and it is inſcribed in the given circle ABCDEF. Which was to be done.

Cor. From this it is manifeſt, that the ſide of the hexagon is equal to the ſtraight line from the center, that is, to the ſemidiameter of the circle.

And if thro' the points A, B, C, D, E, F there be drawn ſtraight lines touching the circle, an equilateral and equiangular hexagon ſhall be deſcribed about it, which may be demonſtrated from what has been ſaid of the pentagon; and likewiſe a circle may be inſcribed in a given equilateral and equiangular hexagon, and circumſcribed about it, by a method like to that uſed for the pentagon.

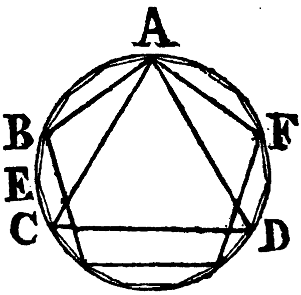
PROP. XVI. PROB.

TO inſcribe an equilateral and equiangular quindecagon in a given circle.

See N.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the ſide of an equilateral triangle inſcribed [a] in the circle, and AB the ſide of an equilateral and equiangular pentagon inſcribed [b] in the ſame; therefore of ſuch equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five; and the circumference AB, which is the fifth part of the whole, contains three; therefore BC their difference contains two of the ſame parts. biſect [c] BC in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD. therefore if the ſtraight lines BE, EC be drawn, and ſtraight lines equal to them be placed [d] around in the whole circle, an equilateral and equiangular quindecagon ſhall be inſcribed in it. Which was to be done.



a. 2. 4.
b. 11. 4.
c. 30. 3.
d. 1. 4.

And in the ſame manner as was done in the pentagon, if thro' the points of diviſion made by inſcribing the quindecagon, ſtraight lines be drawn touching the circle, an equilateral and equiangular quindecagon ſhall be deſcribed about it. and likewiſe, as in the pentagon, a circle may be inſcribed in a given equilateral and equiangular quindecagon, and circumſcribed about it.

THE
ELEMENTS
OF
EUCLID.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is ſaid to be a part of a greater magnitude, when the leſs meaſures the greater, that is, ‘ when the leſs is ‘ contained a certain number of times exactly in the greater.’

II.

A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs, that is, ‘ when the greater con‘ tains the leſs a certain number of times exactly.’

III.

“ Ratio is a mutual relation of two magnitudes of the ſame kind “ to one another, in reſpect of quantity.” See N.

IV.

Magnitudes are ſaid to have a ratio to one another, when the leſs can be multiplied ſo as to exceed the other.

V.

The firſt of four magnitudes is ſaid to have the ſame ratio to the ſecond, which the third has to the fourth, when any equimultiples whatſoever of the firſt and third being taken, and any equimultiples whatſoever of the ſecond and fourth; if the multiple of the firſt be leſs than that of the ſecond, the multiple of the third is alſo leſs than that of the fourth; or, if the multiple of the firſt be equal to that of the ſecond, the multiple of the third is alſo equal to that of the fourth; or, if the multiple of the firſt be greater

Book V. than that of the ſecond, the multiple of the third is alſo greater than that of the fourth.

VI.

Magnitudes which have the ſame ratio are called proportionals. N. B. ‘ When four magnitudes are proportionals, it is uſually ‘ expreſſed by ſaying, the firſt is to the ſecond, as the third to ‘ the fourth.’

VII.

When of the equimultiples of four magnitudes (taken as in the 5th Definition) the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth; and on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

VIII.

“ Analogy, or proportion, is the ſimilitude of ratios.”

IX.

Proportion conſiſts in three terms at leaſt.

X.

When three magnitudes are proportionals, the firſt is ſaid to have to the third the duplicate ratio of that which it has to the ſecond.

XI.

See N. When four magnitudes are continual proportionals, the firſt is ſaid to have to the fourth the Triplicate ratio of that which it has to the ſecond, and ſo on Quadruplicate, &c. increaſing the denomination ſtill by unity, in any number of proportionals.

Definition A, to wit, of Compound ratio.

When there are any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth, and ſo on unto the laſt magnitude.

For example, If A, B, C, D be four magnitudes of the ſame kind, the firſt A is ſaid to have to the laſt D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is ſaid to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B, the ſame ratio which E has to F; and B to C, the ſame ratio that G has to H; and C to D, the ſame that K has to L; then, by this Definition, A is ſaid to have to D the ratio compounded of ratios which are the ſame with the ratios of E to F, G to H, and K to L. and the ſame thing is to be underſtood when it is more briefly expreſſed, by ſaying A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed, if M has to N the ſame ratio which A has to D, then, for ſhortneſs ſake, M is ſaid to have to N, the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one another, as alſo the conſequents to one another.

‘Geometers make uſe of the following technical words to ſignify certain ways of changing either the order or magnitude of proportionals, ſo as that they continue ſtill to be proportionals.’

XIII.

Permutando, or Alternando, by Permutation, or alternately; this word is uſed when there are four proportionals, and it is inferred, that the firſt has the ſame ratio to the third, which the ſecond has to the fourth; or that the firſt is to the third, as the ſecond to the fourth. as is ſhewn in the 16th Prop. of this 5th Book. See N.

XIV.

Invertendo, by Inverſion; when there are four proportionals, and it is inferred, that the ſecond is to the firſt, as the fourth to the third. Prop. B. Book 5th.

XV.

Componendo, by Compoſition; when there are four proportionals, and it is inferred, that the firſt together with the ſecond, is to the ſecond, as the third together with the fourth, is to the fourth. 18th Prop. Book 5th.

XVI.

Dividendo, by Diviſion; when there are four proportionals, and it is inferred, that the Exceſs of the firſt above the ſecond, is to the ſecond, as the Exceſs of the third above the fourth, is to the fourth. 17th Prop. Book 5th.

XVII.

Convertendo, by Converſion; when there are four proportionals, and it is inferred that the firſt is to its Exceſs above the ſecond, as the third to its Exceſs above the fourth. Prop. E. Book 5th.

Book V.

XVIII.

Ex aequali (ſc. diſtantia,) or, ex aequo, from equality of diſtance; when there is any number of magnitudes more than two, and as many others, ſo that they are proportionals when taken two and two of each rank, and it is inferred, that the firſt is to the laſt of the firſt rank of magnitudes, as the firſt is to the laſt of the others. ‘ of this there are the two following kinds, which ‘ ariſe from the different order in which the magnitudes are ‘ taken two and two.’

XIX.

Ex aequali, from equality; this term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo on in order, and the inference is as mentioned in the preceding Definition; whence this is called Ordinate Proportion. It is demonſtrated in 22d Prop. Book 5th.

XX.

Ex aequali, in proportione perturbata, ſeu inordinata, from equality, in perturbate or diſorderly proportion *; this term is uſed when the firſt magnitude is to the ſecond of the firſt rank, as the laſt but one is to the laſt of the ſecond rank; and as the ſecond is to the third of the firſt rank, ſo is the laſt but two to the laſt but one of the ſecond rank; and as the third is to the fourth of the firſt rank, ſo is the third from the laſt to the laſt but two of the ſecond rank; and ſo on in a croſs order. and the inference is as in the 18th Definition. It is demonſtrated in 23d Prop. of Book 5th.

AXIOMS.

I.

EQUIMULTIPLES of the ſame, or of equal magnitudes, are equal to one another.

II.

Thoſe magnitudes of which the ſame, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the ſame multiple of a leſs.

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* 4. Prop. Lib. 2. Archimedis de ſphaera et cylindro.

IV.

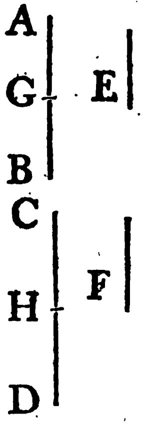
That magnitude of which a multiple is greater than the ſame multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

IF any number of magnitudes be equimultiples of as many, each of each; what multiple ſoever any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each; whatſoever multiple AB is of E, the ſame multiple ſhall AB and CD together be of E and F together.

Becauſe AB is the ſame multiple of E that CD is of F, as many magnitudes as are in AB equal to E, ſo many are there in CD equal to F. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD equal each of them to F. the number therefore of the magnitudes CH, HD ſhall be equal to the number of the others AG, GB. and becauſe AG is equal to E, and CH to F; therefore AG and CH together are equal to [a] E and F together. for the ſame reaſon, becauſe GB is equal to E, and HD to F; GB and HD together are equal to E and F together. Wherefore as many magnitudes as are in AB equal to E, ſo many are there in AB, CD together equal to E and F together. Therefore whatſoever multiple AB is of E, the ſame multiple is AB and CD together of E and F together.



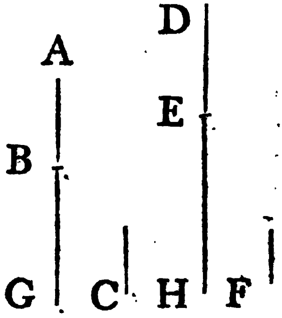
a. Ax. 2. 1.

Therefore if any magnitudes, how many ſoever, be equimultiples of as many, each of each, whatſoever multiple any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other. ‘ for the ſame Demonſtration holds in any number ‘ of magnitudes, which was here applied to two.’ Q. E. D.

PROP. II. THEOR.

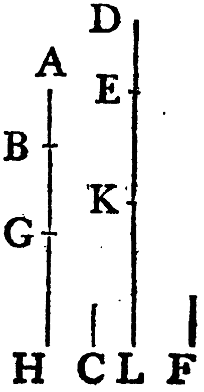
IF the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth; then ſhall the firſt together with the fifth be the ſame multiple of the ſecond, that the third together with the ſixth is of the fourth.

Let AB the firſt be the ſame multiple of C the ſecond, that DE the third is of F the fourth; and BG the fifth the ſame multiple of C the ſecond, that EH the ſixth is of F the fourth. Then is AG the firſt together with the fifth the ſame multiple of C the ſecond, that DH the third together with the ſixth is of F the fourth.



Becauſe AB is the ſame multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F. in like manner, as many as there are in BG equal to C, ſo many are there in EH equal to F. as many then as are in the whole AG equal to C, ſo many are there in the whole DH equal to F. therefore AG is the ſame multiple of C, that DH is of F; that is, AG the firſt and fifth together, is the ſame multiple of the ſecond C, that DH the third and ſixth together is of the fourth F. If therefore the firſt be the ſame multiple, &c. Q. E. D.

Cor. ‘ From this it is plain, that if any ‘ number of magnitudes AB, BG, GH be ‘ multiples of another C; and as many DE, ‘ EK, KL be the ſame multiples of F, each ‘ of each; the whole of the firſt, viz. AH ‘ is the ſame multiple of C, that the whole ‘ of the laſt, viz. DL is of F.’

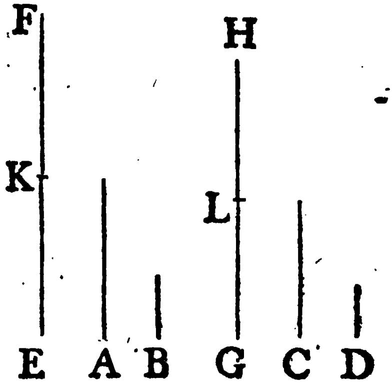


PROP. III. THEOR.

IF the firſt be the ſame multiple of the ſecond, which the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe ſhall be equimultiples the one of the ſecond, and the other of the fourth.

Let A the firſt be the ſame multiple of B the ſecond, that C the third is of D the fourth; and of A, C let the equimultiples EF, GH be taken. then EF is the ſame multiple of B, that GH is of D.

Becauſe EF is the ſame multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as are in GH equal to C. let EF be divided into the magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C. the number therefore of the magnitudes EK, KF, ſhall be equal to the number of the others GL, LH. and becauſe A is the ſame multiple of B, that C is of D, and that EK is equal to A, and GL to C; therefore EK is the ſame multiple of B, that GL is of D. for the ſame reaſon KF is the ſame multiple of B, that LH is of D; and ſo, if there be more parts in EF, GH equal to A, C. becauſe therefore the firſt EK is the ſame multiple of the ſecond B, which the third GL is of the fourth D, and that the fifth KF is the ſame multiple of the ſecond B, which the ſixth LH is of the fourth D; EF the firſt together with the fifth is the ſame multiple [a] of the ſecond B, which GH the third together with the ſixth is of the fourth D. If therefore the firſt, &c. Q. E. D.



a. 2. 5.

PROP. IV. THEOR.

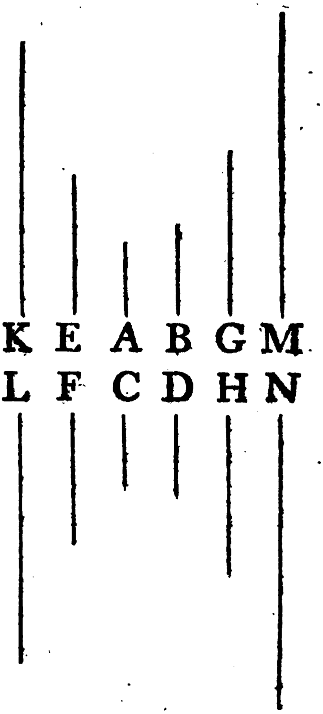
See N.

IF the firſt of four magnitudes has the ſame ratio to the ſecond which the third has to the fourth; then any equimultiples whatever of the firſt and third ſhall have the ſame ratio to any equimultiples of the ſecond and fourth, viz. ‘ the equimultiple of the firſt ſhall ‘ have the ſame ratio to that of the ſecond, which the ‘ equimultiple of the third has to that of the fourth.’

Let A the firſt have to B the ſecond, the ſame ratio which the third C has to the fourth D; and of A and C let there be taken any equimultiples whatever E, F; and of B and D any equimultiples whatever G, H. then E has the ſame ratio to G, which F has to H.

Take of E and F any equimultiples whatever K, L, and of G, H, any equimultiples whatever M, N. then becauſe E is the ſame multiple of A, that F is of C; and of E and F have been taken equimultiples K, L; therefore K is the ſame multiple of A, that L is of C [a]. for the ſame reaſon M is the ſame multiple of B, that N is of D. and becauſe as A is to B, ſo is C to D [b], and of A and C have been taken certain equimultiples K, L; and of B and D have been taken certain equimultiples M, N; if therefore K be greater than M, L is greater than N; and if equal, equal; if leſs, leſs [c]. And K, L are any equimultiples whatever of E, F; and M, N any whatever of G, H. as therefore E is to G, ſo is [c] F to H. Therefore if the firſt, &c. Q. E. D.

a. 3. 5.
b. Hypoth.
c. 5. Def. 5.



Cor. Likewiſe if the firſt has the ſame ratio to the ſecond, which the third has to the fourth, then alſo any equimultiples whatever of the firſt and third have the ſame ratio to the ſecond and fourth. Book V. and in like manner the firſt and the third have the ſame ratio to any equimultiples whatever of the ſecond and fourth.

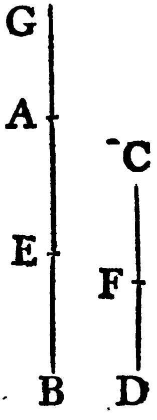
Let A the firſt have to B the ſecond, the ſame ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; then E is to B, as F to D.

Take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H; then it may be demonſtrated, as before, that K is the ſame multiple of A, that L is of C. and becauſe A is to B, as C is to D, and of A and C certain equimultiples have been taken, viz. K and L; and of B and D certain equimultiples G, H; therefore if K be greater than G, L is greater than H; and if equal, equal; if leſs, leſs [c]. and K, L are any equimultiples of c. 5. Def. 5. E, F, and G, H any whatever of B, D; as therefore E is to B, ſo is F to D. and in the ſame way the other caſe is demonſtrated.

PROP. V. THEOR.

IF one magnitude be the ſame multiple of another, which a magnitude taken from the firſt is of a magnitude taken from the other; the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole. See N.

Let the magnitude AB be the ſame multiple of CD, that AE taken from the firſt, is of CF taken from the other; the remainder EB ſhall be the ſame multiple of the remainder FD, that the whole AB is of the whole CD.

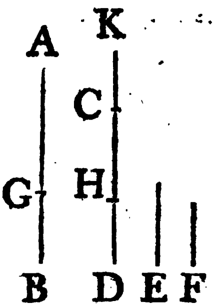


Take AG the ſame multiple of FD, that AE is of CF. therefore AE is [a] the ſame multiple of CF, that EG is of CD. but AE, by the hypotheſis, is the ſame multiple of CF, that AB is of CD. a. 1. 5. therefore EG is the ſame multiple of CD that AB is of CD; wherefore EG is equal to AB [b]. take b. 1. Ax. 5. from them the common magnitude AE; the remainder AG is equal to the remainder EB. Wherefore ſince AE is the ſame multiple of CF, that AG is of FD, and that AG is equal to EB; therefore AE is the ſame multiple of CF, that EB is of FD. but AE is the ſame multiple of CF, that AB is of CD; therefore EB is the ſame multiple of FD, that AB is of CD. Therefore if one magnitude, &c. Q. E. D.

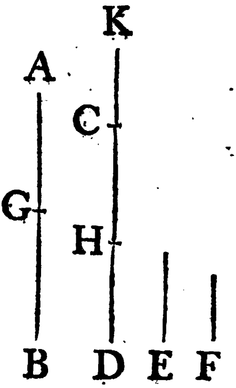
PROP. IV. THEOR.

See N. IF two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F, and AG, CH taken from the firſt two be equimultiples of the ſame E, F; the remainders GB, HD are either equal to E, F, or equimultiples of them.



Firſt, Let GB be equal to E; HD is equal to F. make CK equal to F; and becauſe AG is the ſame multiple of E, that CH is of F, and that GB is equal to E, and CK to F; therefore AB is the ſame multiple of E, that KH is of F. But AB, by the hypotheſis, is the ſame multiple of E that CD is of F; therefore KH is the ſame multiple of F, that CD is of F; wherefore a. 1. Ax. 5. KH is equal to CD [a]. take away the common magnitude CH, then the remainder KC is equal to the remainder HD. but KC is equal to F, HD therefore is equal to F.



But let GB be a multiple E; then HD is the ſame multiple of F. Make CK the ſame multiple of F, that GB is of E. and becauſe AG is the ſame multiple of E, that CH is of F, and GB the ſame multiple of E, that CK is of F, therefore AB is the ſame multiple of E, that KH b. 2. 5. is of F [b]. but AB is the ſame multiple of E, that CD is of F; therefore KH is the ſame multiple of F, that CD is of it; wherefore KH is equal to CD [a]. take away CH from both, therefore the remainder KC is equal to the remainder HD. and becauſe GB is the ſame multiple of E, that KC is of F, and that KC is equal to HD; therefore HD is the ſame multiple of F, that GB is of E. If therefore two magnitudes, &c. Q. E. D.

PROP. A. THEOR.

See N.

IF the firſt of four magnitudes has to the ſecond, the ſame ratio which the third has to the fourth; then if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and if equal, equal; if leſs, leſs.

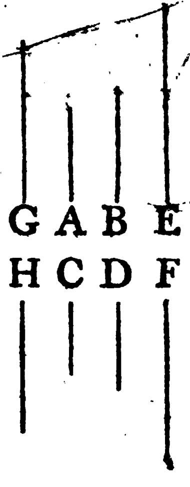
Take any equimultiples of each of them, as the doubles of each. then by Def. 5th of this Book, if the double of the firſt be greater than the double of the ſecond, the double of the third is greater than the double of the fourth. but if the firſt be greater than the ſecond, the double of the firſt is greater than the double of the ſecond. wherefore alſo the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth. in like manner, if the firſt be equal to the ſecond, or leſs than it, the third can be proved to be equal to the fourth, or leſs than it. Therefore if the firſt, &c. Q. E. D.

PROP. B. THEOR.

See N.

IF four magnitudes are proportionals, they are proportionals alſo when taken inverſely.

If the magnitude A be to B, as C is to D, then alſo inverſely B is to A, as D to C.



Take of B and D any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H. Firſt, Let E be greater than G, then G is leſs than E; and becauſe A is to B, as C is to D, and of A and C the firſt and third, G and H are equimultiples; and of B and D the ſecond and fourth, E and F are equimultiples; and that G is leſs than E, H is alſo [a] leſs than F; that is, F is greater than H. if therefore E be greater than G, F is greater than H. in like manner, if E be equal to G, F may be ſhewn to be equal to H; and if leſs, leſs. and E, F are any equimultiples whatever of B and D, and G, H

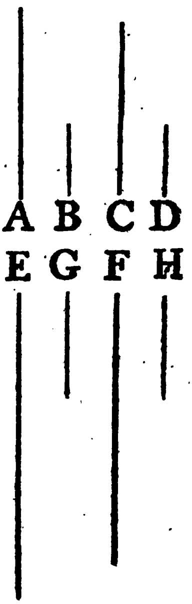
a. 5. Def. 5.

any whatever of A and C. Therefore as B is to A, ſo is D to C. If then four magnitudes, &c. Q. E. D.

PROP. C. THEOR.

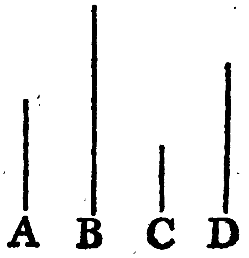
See N. IF the firſt be the ſame multiple of the ſecond, or the ſame part of it, that the third is of the fourth; the firſt is to the ſecond, as the third is to the fourth.

Let the firſt A be the ſame multiple of B the ſecond, that C the third is of the fourth D. A is to B, as C is to D.



Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H. then becauſe A is the ſame multiple of B that C is of D; and that E is the ſame multiple of A, that F is of C; E is the ſame multiple of B, that
a. 3. 5. F is of D [a]; therefore E and F are the ſame multiples of B and D. but G and H are equimultiples of B and D; therefore if E be a greater multiple of B, than G is; F is a greater multiple of D, than H is of D; that is, if E be greater than G, F is greater than H. in like manner, if E be equal to G, or leſs; F is equal to H, or leſs than it. But E, F are equimultiples, any whatever, of A, C, and G, H any equimultiples whatever of
b. 5. Def. 5. B, D. Therefore A is to B, as C is to D [b].

Next, Let the firſt A be the ſame part of the ſecond B, that the third C is of the fourth D. A is to B, as C is to D. for B is the ſame multiple of A, that D is of C; wherefore by the preceding caſe B is to A, as D is to
c. B. 5. C; and inverſely [c] A is to B, as C is to D. Therefore if the firſt be the ſame multiple, &c. Q. E. D.

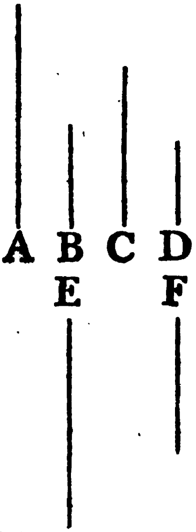


PROP. D. THEOR.

See N.

IF the firſt be to the ſecond as the third to the fourth, and if the firſt be a multiple, or part of the ſecond; the third is the ſame multiple, or the ſame part of the fourth.

Let A be to B, as C is to D; and firſt let A be a multiple of B; C is the ſame multiple of D.



Take E equal to A, and whatever multiple A or E is of B, make F the ſame multiple of D. then becauſe A is to B, as C is to D; and of B the ſecond and D the fourth equimultiples have been taken E and F; A is to E, as C to F [a]. but A is equal to E, therefore C is equal to F [b]. and F is the ſame multiple of D, that A is of B. Wherefore C is the ſame multiple of D, that A is of B.

a. Cor. 4. 5.

b. A. 5.

See the Figure at the foot of the preceding page.

Next, Let the firſt A be a part of the ſecond B; C the third is the ſame part of the fourth D.

Becauſe A is to B, as C is to D; then, inverſely B is [c] to A, as D to C. but A is a part of B, therefore B is a multiple of A, and, by the preceding caſe, D is the ſame multiple of C; that is, C is the ſame part of D, that A is of B. Therefore if the firſt, &c. Q. E. D.

c. B. 5.

PROP. VII. THEOR.

EQUAL magnitudes have the ſame ratio to the ſame magnitude; and the ſame has the ſame ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other. A and B have each of them the ſame ratio to C. and C has the ſame ratio to each of the magnitudes A and B.

Take of A and B any equimultiples whatever D and E, and of

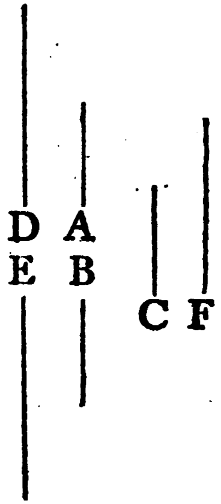
C any multiple whatever F. then becauſe D is the ſame multiple of A, that E is of B, and that A is equal to

a. 1. Ax. 5.

B; D is [a] equal to E. therefore if D be greater than F, E is greater than F; and if equal, equal; if leſs, leſs. and D, E are any equimultiples of A, B, and F is any multiple of C.

b. 5. Def. 5.

Therefore [b] as A is to C, ſo is B to C.



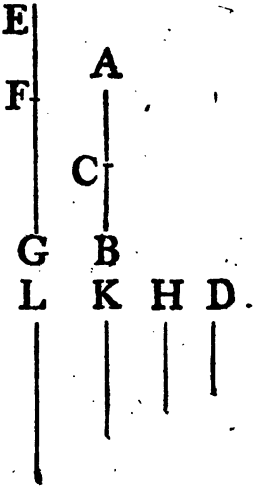
Likewiſe C has the ſame ratio to A that it has to B. for, having made the ſame conſtruction, D may in like manner be ſhewn equal to E. therefore if F be greater than D, it is likewiſe greater than E; and if equal, equal; if leſs, leſs. and F is any multiple whatever of C, and D, E, are any equimultiples whatever of A, B. Therefore C is to A, as C is to B[b]. Therefore equal magnitudes, &c. Q. E. D.

PROP. VIII. THEOR.

See N.

Of unequal magnitudes the greater has a greater ratio to the ſame than the leſs has. and the ſame magnitude has a greater ratio to the leſs than it has to the greater.

Let AB, BC be unequal magnitudes of which AB is the greater, and let D be any magnitude whatever. AB has a greater ratio to D than BC to D. and D has a greater ratio to BC than unto AB.



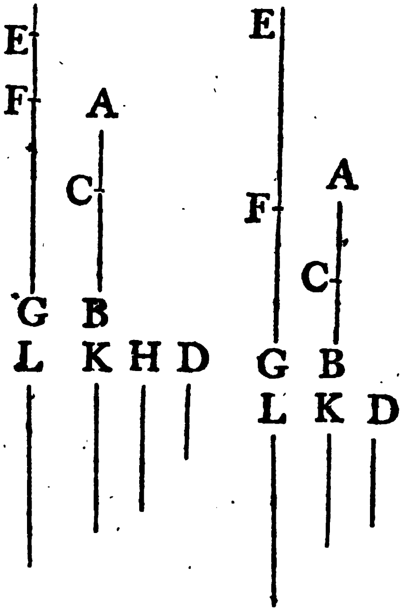
If the magnitude which is not the greater of the two AC, CB, be not leſs than D, take EF, FG the doubles of AC, CB, as in Fig. 1. but if that which is not the greater of the two AC, CB be leſs than D (as in Fig. 2. and 3.) this magnitude can be multiplied ſo as to become greater than D, whether it be AC or CB. Let it be multiplied until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the ſame multiple of CB.

therefore EF and FG are each of them greater than D. and in every one of the caſes take H the double of D, K its triple, and ſo on, till the multiple of D be that which firſt becomes greater than FG. let L be that multiple of D which is firſt greater than FG, and K the multiple of D which is next leſs than L.

Then becauſe L is the multiple of D which is the firſt that becomes greater than FG, the next preceding multiple K is not greater than FG; that is, FG is not leſs than K. and ſince EF is the ſame multiple of AC, that FG is of CB; FG is the ſame multiple of CB, that EG is of AB [a]; wherefore EG and FG are equimultiples of AB and CB. and it was ſhewn that FG was not leſs than K, and, by the conſtruction, EF is greater than D; therefore the whole EG is greater than K and D together. but K together with D is equal to L; therefore EG is greater than L; but FG is not greater than L; and EG, FG are equimultiples of AB, BC, and L is a multiple of D; therefore [b] AB has to D a greater ratio than BC has to D.

a. 1. 5.

b. 7. Def. 5.



Alſo D has to BC a greater ratio than it has to AB, for, having made the ſame conſtruction, it may be ſhewn, in like manner, that L is greater than FG, but that it is not greater than EG. and L is a multiple of D; and FG, EG are equimultiples of CB, AB. Therefore D has to CB a greater ratio [b] than it has to AB. Wherefore of unequal magnitudes, &c. Q. E. D.

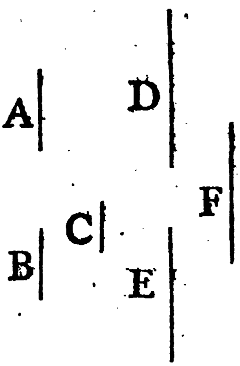
PROP. IX. THEOR.

See N.

MAGNITUDES which have the ſame ratio to the ſame magnitude are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio are equal to one another.

Let A, B have each of them the ſame ratio to C; A is equal to B. for if they are not equal, one of them is greater than the other; let A be the greater; then, by what was ſhewn in the preceding Propoſition, there are ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let ſuch multiples be taken, and let D, E, be the equimultiples of A, B, and F the multiple of C ſo that D may be greater than F, and E not greater than F. but becauſe A is to C, as B is to C, and of A, B are taken equimultiples D, E, and of C is taken a multiple F; and that D is greater than F; E ſhall alſo be greater than F [a]; but E is not greater than F, which is impoſſible. A therefore and B are not unequal; that is, they are equal.

a. 5. Def. 5.



Next, Let C have the ſame ratio to each of the magnitudes A and B; A is equal to B. for if they are not, one of them is greater than the other; let A be the greater, therefore, as was ſhewn in Prop. 8th, there is ſome multiple F of C, and ſome equimultiples E and D of B and A ſuch, that F is greater than E, and not greater than D. but becauſe C is to B, as C is to A, and that F the multiple of the firſt is greater than E the multiple of the ſecond; F the multiple of the third is greater than D the multiple of the fourth [a]. but F is not greater than D, which is impoſſible. Therefore A is equal to B. Wherefore magnitudes which, &c. Q. E. D.

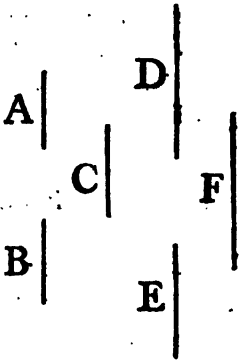
PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has unto the ſame magnitude is the greater of the two. and that magnitude to which the ſame has a greater ratio than it has unto another magnitude is the leſſer of the two.

See N.

Let A have to C a greater ratio than B has to C; A is greater than B. for becauſe A has a greater ratio to C, than B has to C, there are [a] ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it. let them be taken, and let D, E be equimultiples of A, B, and F a multiple of C ſuch, that D is greater than F, but E is not greater than F. therefore D is greater than E. and becauſe D and E are equimultiples of A and B, and D is greater than E; therefore A is [b] greater than B.

a. 7. Def. 5.
b. 4. Ax. 5.



Next, Let C have a greater ratio to B than it has to A; B is leſs than A. for [a] there is ſome multiple F of C, and ſome equimultiples E and D of B and A ſuch, that F is greater than E, but is not greater than D. E therefore is leſs than D; and becauſe E and D are equimultiples of B and A, therefore B is [b] leſs than A. That magnitude therefore, &c. Q. E. D.

PROP. XI. THEOR.

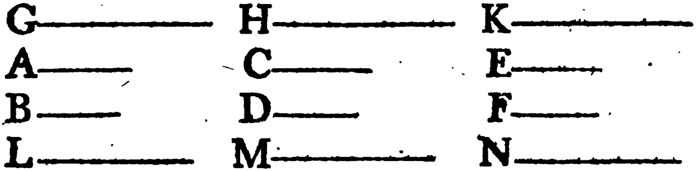
RATIOS that are the ſame to the ſame ratio, are the ſame to one another.

Let A be to B, as C is to D; and as C to D, ſo let E be to F; A is to B, as E to F.

Take of A, C, E any equimultiples whatever G, H, K; and of B, D, F any equimultiples whatever L, M, N. Therefore ſince A is to B, as C to D, and of A, C are taken equimultiples G,

a. 5. Def. 5.

H; and L, M of B, D; if G be greater than L, H is greater than M; and if equal, equal; and if leſs, leſs [a]. Again, becauſe C is to D, as E is to F, and H, K are taken equimultiples of C, E; and M, N of D, F; if H be greater than M, K is greater than N; and if equal, equal; and if leſs, leſs. but if G be greater than L, it has



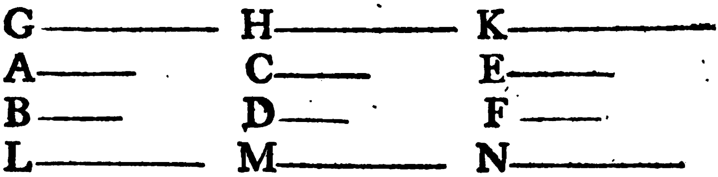
been ſhewn that H is greater than M; and if equal, equal; and if leſs, leſs; therefore if G be greater than L, K is greater than N; and if equal, equal; and if leſs, leſs. and G, K, are any equimultiples whatever of A, E; and L, N any whatever of B, F. Therefore as A is to B, ſo is E to F [a]. Wherefore ratios that, &c. Q. E. D.

PROP. XII. THEOR.

IF any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.

Let any number of magnitudes A, B; C, D; E, F, be proportionals; that is, as A is to B, ſo C to D, and E to F. as A is to B, ſo ſhall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H, K; and of



B, D, F any equimultiples whatever L, M, N. then becauſe A is to B, as C is to D, and as E to F; and that G, H, K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; if G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if leſs, leſs [a]. Wherefore if G be greater than L, then G, H, K together are greater than L, M, N together; and if equal, equal; and if leſs, leſs. and G, and G, H, K together are any equimultiples of A, and A, C, E together, becauſe if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the ſame multiple is the whole of the whole [b]. for the ſame reaſon L, and L, M, N are any equimultiples of B, and B, D, F. as therefore A is to B, ſo are A, C, E together to B, D, F together. Wherefore if any number, &c. Q. E. D.

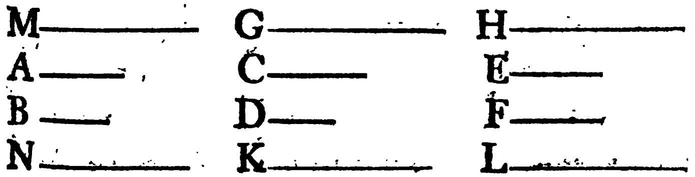
a. 5. Def. 5.
b. 1. 5.

PROP. XIII. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth. See N.

Let A the firſt have the ſame ratio to B the ſecond which C the third has to D the fourth, but C the third to D the fourth a greater ratio than E the fifth to F the ſixth. alſo the firſt A ſhall have to the ſecond B a greater ratio than the fifth E to the ſixth F.

Becauſe C has a greater ratio to D, than E to F, there are ſome equimultiples of C and E, and ſome of D and F ſuch, that the multiple of C is greater than the multiple of D, but the multiple of E is



not greater than the multiple of F [a]. let ſuch be taken, and of C, E let G, H be equimultiples, and K, L equimultiples of D, F ſo that G be greater than K, but H not greater than L; and whatever multiple G is of C take M the ſame multiple of A; and what multiple K is of D, take N the ſame multiple of B. then becauſe A is to B, as C to D, and of A and C, M and G are equimultiples, and of

a. 7. Def. 5.

B and D, N and K are equimultiples; if M be greater than N,
G is greater than K; and if equal, equal; and if leſs, leſs[b]; but
b. 5. Def. 5. G is greater than K, therefore M is greater than N. but H is not
greater than L; and M, H are equimultiples of A, E; and N, L
equimultiples of B, F. Therefore A has a greater ratio to B, than
a. 7. Def. 5. E has to F[a]. Wherefore if the firſt, &c. Q. E. D.

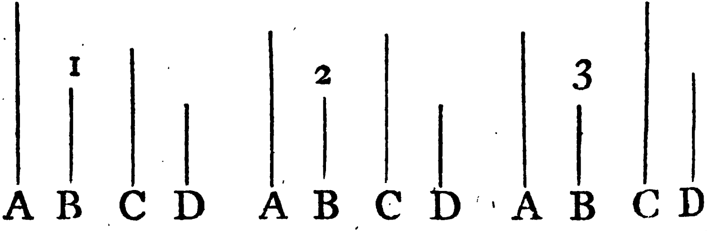
Cor. And if the firſt has a greater ratio to the ſecond, than the third has to the fourth; but the third the ſame ratio to the fourth, which the fifth has to the ſixth; it may be demonſtrated in like manner that the firſt has a greater ratio to the ſecond than the fifth has to the ſixth.

PROP. XIV. THEOR.

See N. **If the firſt has to the ſecond the ſame ratio, which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.**

Let the firſt A have to the ſecond B the ſame ratio, which the third C has to the fourth D; if A be greater than C, B is greater than D.

Becauſe A is greater than C, and B is any other magnitude, A
a. 8. 5. has to B a greater ratio than C to B[a]. but as A is to B, ſo is C to



b. 13. 5. D; therefore alſo C has to D a greater ratio than C has to B[b]. but of
two magnitudes, that to which the ſame has the greater ratio is the
c. 10. 5. leſſer[c]. wherefore D is leſs than B; that is, B is greater than D.

Secondly, If A be equal to C, B is equal D. for A is to B, as
d. 9. 5. C, that is A, to D; B therefore is equal to D[d].

Thirdly, If A be leſs than C, B ſhall be leſs than D. for C is greater than A, and becauſe C is to D, as A is to B, D is greater than B by the firſt caſe; wherefore B is leſs than D. Therefore if the firſt, &c. Q. E. D.

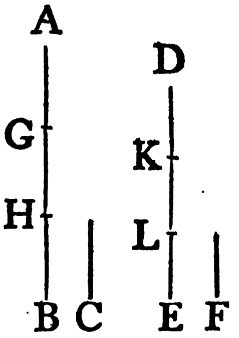
PROP. XV. THEOR.

MAGNITUDES have the ſame ratio to one another which their equimultiples have.

Let AB be the ſame multiple of C that DE is of F. C is to F, as AB to DE.

Becauſe AB is the ſame multiple of C that DE is of F, there are as many magnitudes in AB equal to C, as there are in DE equal to F. Let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE. then the number of the firſt AG, GH, HB ſhall be equal to the number of the laſt DK, KL, LE. and becauſe AG, GH, HB are all equal, and that DK, KL, LE are alſo equal to one another; therefore AG is to DK, as GH to KL, and as HB to LE[a]. and as one of the antecedents to its conſequent, ſo are all the antecedents together to all the conſequents together[b]; wherefore as AG is to DK, ſo is AB to DE. but AG is equal to C, and DK to F. therefore as C is to F, ſo is AB to DE. Therefore magnitudes, &c. Q. E. D.

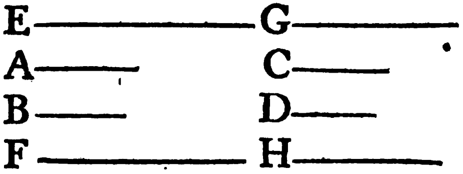
a. 7. 5.
b. 12. 5.



PROP. XVI. THEOR.

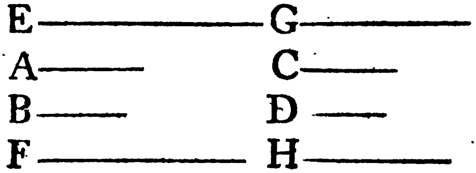
IF four magnitudes of the ſame kind be proportionals, they ſhall alſo be proportionals when taken alternately.

Let the four magnitudes A, B, C, D be proportionals, viz. as A to B, ſo C to D. they ſhall alſo be proportionals when taken alternately; that is, A is to C, as B to D.



Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever

G and H. and becauſe E is the ſame multiple of A, that F is of
B, and that magnitudes have the ſame ratio to one another which
a. 15. 5. their equimultiples have [a]; therefore A is to B, as E is to F. but
b. 11. 5. as A is to B, ſo is C to D. wherefore as C is to D, ſo [b] is E to F.
again, becauſe G, H are equimultiples of C, D, as C is to D, ſo is G to H [a]; but as C is to D, ſo is E to F. Wherefore as E is to F, ſo is G to H [b]. But when four magnitudes are proportionals, if the firſt be
greater than the third, the ſecond ſhall be greater than the fourth;
c. 14. 5. and if equal, equal; if leſs, leſs [c]. Wherefore if E be greater
than G, F likewiſe is greater than H; and if equal, equal; if leſs,
leſs. and E, F are any equimultiples whatever of A, B; and G,
d. 5. Def. 5. H any whatever of C, D. Therefore A is to C, as B to D [d]. If
then four magnitudes, &c. Q. E. D.



PROP. XVII. THEOR.

See N. IF magnitudes taken jointly be proportionals, they ſhall alſo be proportionals when taken ſeparately, that is, if two magnitudes together have to one of them, the ſame ratio which two others have to one of theſe, the remaining one of the firſt two ſhall have to the other, the ſame ratio which the remaining one of the laſt two has to the other of theſe.

Let AB, BE; CD, DF be the magnitudes taken jointly which are proportionals; that is, as AB to BE, ſo is CD to DF; they ſhall alſo be proportionals taken ſeparately, viz. as AE to EB, ſo CF to FD.

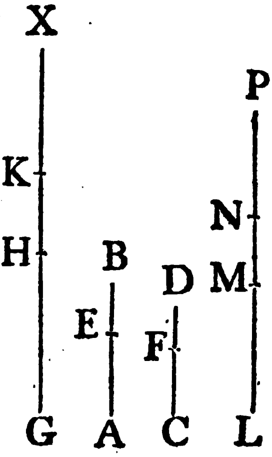
Take of AE, EB, CF, FD any equimultiples whatever GH, HK,
LM, MN; and again, of EB, FD, take any equimultiples whatever
KX, NP. and becauſe GH is the ſame multiple of AE that HK is
a. 1. 5. of EB, therefore GH is the ſame multiple [a] of AE, that GK is of
AB. but GH is the ſame multiple of AE, that LM is of CF; wherefore GK is the ſame multiple of AB, that LM is of CF. Again, be-

cauſe LM is the ſame multiple of CF that MN is of FD; therefore LM is the ſame multiple [a] of CF, that LN is of CD. but LM was ſhewn to be the ſame multiple of CF, that GK is of AB; GK therefore is the ſame multiple of AB, that LN is of CD; that is, GK, LN are equimultiples of AB, CD. Next, becauſe HK is the ſame multiple of EB, that MN is of FD; and that KX is alſo the ſame multiple of EB, that NP is of FD; therefore HX is the ſame multiple [b] of EB that MP is of FD. And becauſe AB is to BE, as CD is to DF, and that of AB and CD, GK and LN are equimultiples, and of EB and FD, HX and MP are equimultiples; if GK be greater than HX, then LN is greater than MP; and if equal, equal; and if leſs, leſs [c]. but if GH be greater than KX, by adding the common part HK to both, GK is greater than HX; wherefore alſo LN is greater than MP; and by taking away MN from both, LM is greater than NP. therefore if GH be greater than KX, LM is greater than NP. In like manner it may be demonſtrated, that if GH be equal to KX, LM likewiſe is equal to NP; and if leſs, leſs. and GH, LM are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD. Therefore [c] as AE is to EB, ſo is CF to FD. If then magnitudes, &c. Q. E. D.

a. 1. 5.

b. 2. 5.

c. 5. Def. 5.



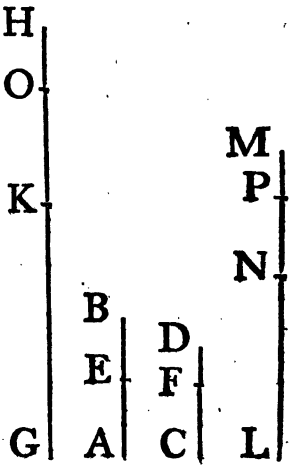
PROP. XVIII. THEOR.

See N.

IF magnitudes taken ſeparately be proportionals, they ſhall alſo be proportionals when taken jointly, that is, if the firſt be to the ſecond, as the third to the fourth, the firſt and ſecond together ſhall be to the ſecond, as the third and fourth together to the fourth.

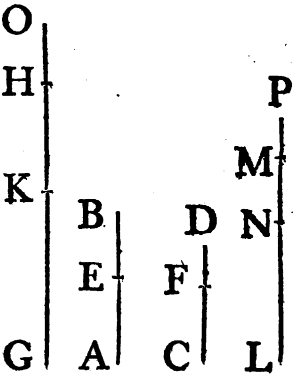
Let AE, EB, CF, FD be proportionals; that is, as AE to EB, ſo is CF to FD; they ſhall alſo be proportionals when taken jointly; that is, as AB to BE, ſo CD to DF.

Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN; and again of BE, DF take any whatever equimultiples KO, NP. and becauſe KO, NP are equimultiples of BE, DF; and that KH, NM are equimultiples likewiſe of BE, DF, if KO the multiple of BE be greater than KH which is a multiple of the ſame BE, NP likewiſe the multiple of DF ſhall be greater than NM the multiple of the ſame DF; and if KO be equal to KH, NP ſhall be equal to NM; and if leſs, leſs.



Firſt, Let KO not be greater than KH, therefore NP is not greater than NM. and becauſe GH, HK are equimultiples of AB, BE, and that AB is greater than BE, therefore GH is greater [a] than HK; but KO is not greater than KH, wherefore GH is greater than KO. In like manner it may be ſhewn that LM is greater than NP. Therefore if KO be not greater than KH, then GH the multiple of AB is always greater than KO the multiple of BE; and likewiſe LM the multiple of CD greater than NP the multiple of DF.

a. 3. Ax. 5.



Next, Let KO be greater than KH; therefore, as has been ſhewn, NP is greater than NM. and becauſe the whole GH is the ſame multiple of the whole AB, that HK is of BE, the remainder GK is the ſame multiple of the remainder AE that GH is of AB [b], which is the ſame that LM is of CD. In like manner, becauſe LM is the ſame multiple of CD, that MN is of DF, the remainder LN is the ſame multiple of the remainder CF, that the whole LM is of the whole CD [b]. but it was ſhewn that LM is the ſame multiple of CD that GK is of AE; therefore GK is the ſame multiple of AE that LN is of CF; that is, GK, LN are equimultiples of AE, CF. and becauſe KO, NP are equimultiples of BE, DF, if from KO, NP there be taken KH, NM, which are likewiſe equimultiples of BE, DF, the remainders HO, MP are either equal to BE, DF, or equimultiples of them [c]. Firſt, Let HO, MP be equal to BE, DF;

b. 5. 5.

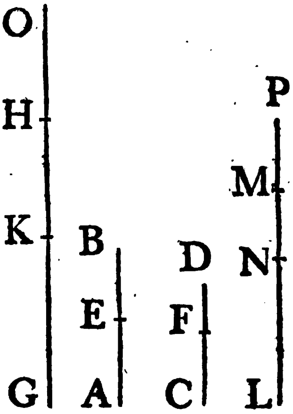
c. 6. 5.

and becauſe AE is to EB, as CF to FD, and that GK, LN are equimultiples of AE, CF; GK ſhall be to EB, as LN to FD [d]. but HO is equal to EB, and MP to FD; wherefore GK is to HO, as LN to MP. If therefore GK be greater than HO, LN is greater than MP; and if equal, equal; and if leſs, leſs [e].

d. Cor. 4. 5.

e. A. 5.

But let HO, MP be equimultiples of EB, FD; and becauſe AE is to EB, as CF to FD, and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; if GK be greater than HO, LN is greater than MP; and if equal, equal; and if leſs, leſs [f]; which was likewiſe ſhewn in the preceding caſe. If therefore GH be greater than KO, taking KH from both, GK is greater than HO; wherefore alſo LN is greater than MP; and conſequently, adding NM to both, LM is greater than NP. therefore if GH be greater than KO, LM is greater than NP. In like manner it may be ſhewn that if GH be equal to KO, LM is equal to NP; and if leſs, leſs. And in the caſe in which KO is not greater than KH, it has been ſhewn that GH is always greater than KO, and likewiſe LM than NP. but GH, LM are any equimultiples of AB, CD, and KO, NP are any whatever of BE, DF; therefore [f] as AB is to BE, ſo is CD to DF. If then magnitudes, &c. Q. E. D.



f. 5. Def. 5.

PROP. XIX. THEOR.

IF a whole magnitude be to a whole, as a magnitude taken from the firſt is to a magnitude taken from the other; the remainder ſhall be to the remainder as the whole to the whole.

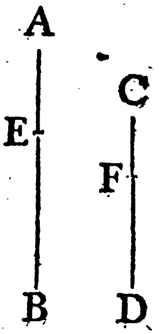
See N.

Let the whole AB be to the whole CD, as AE a magnitude taken from AB to CF a magnitude taken from CD; the remainder EB ſhall be to the remainder FD, as the whole AB, to the whole CD.

Book V.

a. 16. 5.
b. 17. 5.

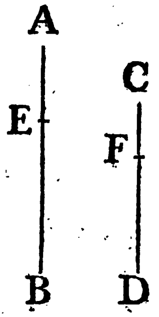
Becauſe AB is to CD, as AE to CF; likewiſe, alternately [a] BA is to AE, as DC to CF. and becauſe if magnitudes taken jointly be proportionals, they are alſo proportionals [b] when taken ſeparately; therefore as BE is to EA, ſo is DF to FC; and alternately, as BE is to DF, ſo is EA to FC. but as AE to CF, ſo, by the Hypotheſis, is AB to CD; therefore alſo BE the remainder ſhall be to the remainder DF, as the whole AB to the whole CD. Wherefore if the whole, &c. Q. E. D.



Cor. If the whole be to the whole, as a magnitude taken from the firſt is to a magnitude taken from the other; the remainder likewiſe is to the remainder, as the magnitude taken from the firſt to that taken from the other. the Demonſtration is contained in the preceding.

PROP. E. THEOR.

If four magnitudes be proportionals, they are alſo proportionals by converſion, that is, the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth.



Let AB be to BE, as CD to DF; then BA is to AE, as DC to CF.

a. 17. 5.
b. B. 5.
c. 18. 5.

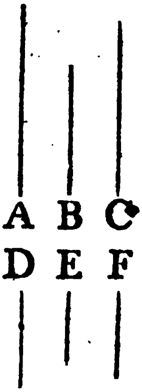
Becauſe AB is to BE, as CD to DF, by diviſion [a], AE is to EB, as CF to FD; and by inverſion [b], BE is to EA, as DF to FC. Wherefore, by Compoſition [c], BA is to AE, as DC is to CF. If therefore four, &c. Q. E. D.

PROP. XX. THEOR.

See N.

If there be three magnitudes, and other three, which taken two and two have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let A, B, C be three magnitudes, and D, E, F other three, which taken two and two have the ſame ratio, viz. as A is to B, ſo is D to E; and as B to C, ſo is E to F. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſs.

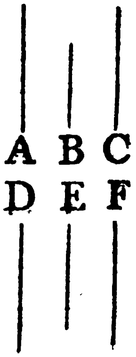


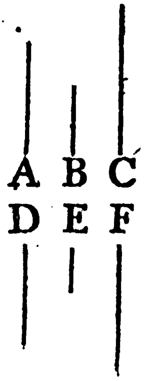
Becauſe A is greater than C, and B is any other magnitude, and that the greater has to the ſame magnitude a greater ratio than the leſs has to it [a]; therefore A has to B a greater ratio than C has to B. but as D is to E, ſo is A to B; therefore [b] D has to E a greater ratio than C to B. and becauſe B is to C, as E to F, by inverſion, C is to B, as F is to E; and D was ſhewn to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E [c]. but the magnitude which has a greater ratio than another to the ſame magnitude, is the greater of the two [d], D is therefore greater than F.

a. 8. 5.
b. 13. 5.
c. Cor. 13. 5.
d. 10. 5.

Secondly, Let A be equal to C; D ſhall be equal to F, becauſe A and C are equal to one another, A is to B, as C is to B [e]. but A is to B, as D to E; and C is to B, as F to E; wherefore D is to E, as F to E [f]; and therefore D is equal to F [g].

e. 7. 5.
f. 11. 5.
g. 9. 5.



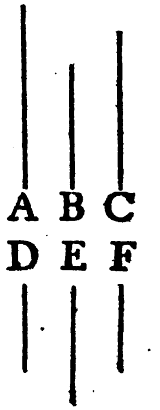


Next, Let A be leſs than C; D ſhall be leſs than F. for C is greater than A, and, as was ſhewn in the firſt caſe, C is to B, as F to E, and in like manner B is to A, as E to D; therefore F is greater than D, by the firſt caſe; and therefore D is leſs than F. Therefore if there be three, &c. Q. E. D.

PROP. XXI. THEOR.

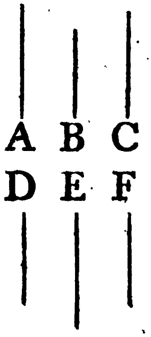
IF there be three magnitudes, and other three, which have the ſame ratio taken two and two, but in a croſs order; if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

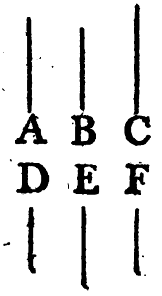
Let A, B, C be three magnitudes, and D, E, F other three which have the ſame ratio taken two and two, but in a croſs order, viz. as A is to B, ſo is E to F, and as B is to C, ſo is D to E. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſs.



Becauſe A is greater than C, and B is any other
a. 8. 5. magnitude, A has to B, a greater ratio [a] than C
b. 13. 5. has to B. but as E to F, ſo is A to B; therefore [b]
E has to F a greater ratio than C to B. and becauſe B is to C, as D to E, by inverſion, C is to B, as E to D. and E was ſhewn to have to F a greater ratio than C to B; therefore E has to F a
c. Cor. 13. 5. greater ratio than E to D [c]. but the magnitude to which the ſame has a greater ratio than it has to
d. 10. 5. another, is the leſſer of the two [d]. F therefore is leſs than D; that is, D is greater than F.

Secondly, Let A be equal to C; D ſhall be equal to F. Becauſe
e. 7. 5. A and C are equal, A is [e] to B, as C is to B. but A is to B, as E to F; and C is to B, as E to D;
f. 11. 5. wherefore E is to F, as E to D [f];
g. 9. 5. and therefore D is equal to F [g].



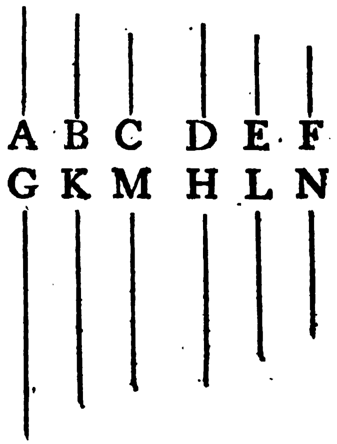


Next, Let A be leſs than C; D ſhall be leſs than F. for C is greater than A, and, as was ſhewn, C is to B, as E to D, and in like manner B is to A, as F to E; therefore F is greater than D, by caſe firſt; and therefore D is leſs than F. Therefore if there be three, &c. Q. E. D.

PROP. XXII. THEOR.

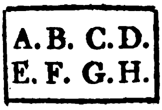
See N. IF there be any number of magnitudes, and as many others, which taken two and two in order have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the other has to the laſt. N. B. *This is uſually cited by the words "ex aequali, or, ex aequo."*

Firſt, Let there be three magnitudes A, B, C, and as many others D, E, F, which taken two and two have the ſame ratio, that is ſuch that A is to B, as D to E; and as B is to C, ſo is E to F. A ſhall be to C, as D to F.



Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N. then becauſe A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; as G is to K, ſo is [a] H to L. for the ſame reaſon K is to M, as L to N. and becauſe there are three magnitudes G, K, M, and other three H, L, N, which two and two have the ſame ratio; if G be greater than M, H is greater than N; and if equal, equal; and if leſs, leſs [b]. and G, H are any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F. therefore [c] as A is to C, ſo is D to F.

a. 4. 5.
b. 20. 5.
c. 5. Def. 5.



Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which two and two have the ſame ratio, viz. as A is to B, ſo is E to F; and as B to C, ſo F to G; and as C to D, ſo G to H. A ſhall be to D, as E to H.

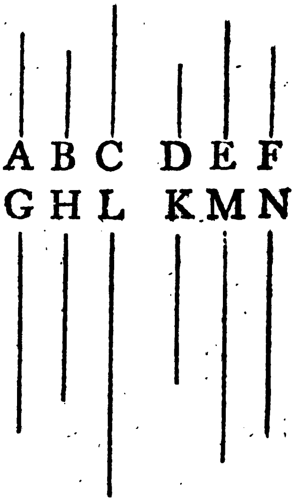
Becauſe A, B, C are three magnitudes, and E, F, G other three, which taken two and two have the ſame ratio; by the foregoing caſe, A is to C, as E to G. but C is to D, as G is to H; wherefore again, by the firſt caſe, A is to D, as E to H. and ſo on, whatever be the number of magnitudes. Therefore if there be any number, &c. Q. E. D.

PROP. XXIII. THEOR.

See N. IF there be any number of magnitudes, and as many others, which, taken two and two, in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt. N. B. *This is uſually cited by the words "ex aequali in proportione perturbata," or, ex aequo perturbate."*

Firſt, Let there be three magnitudes A, B, C, and other three D, E, F, which taken two and two in a croſs order have the ſame ratio, that is ſuch that A is to B, as E to F; and as B is to C, ſo is D to E. A is to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, F any equimultiples whatever L, M, N. and becauſe G, H
are equimultiples of A, B, and that
magnitudes have the ſame ratio which
a. 15. 5. their equimultiples have[a]; as A is
to B, ſo is G to H. and for the ſame
reaſon, as E is to F, ſo is M to N.
but as A is to B, ſo is E to F; as
b. 11. 5. therefore G is to H, ſo is M to N[b].
and becauſe as B is to C, ſo is D to
E, and that H, K are equimultiples
of B, D, and L, M of C, E; as H
c. 4. 5. is to L, ſo is [c] K to M. and it has
been ſhewn that G is to H, as M
to N. then becauſe there are three
magnitudes, G, H, L, and other three
K, M, N which have the ſame ratio
taken two and two in a croſs order;
if G be greater than L, K is greater
d. 21. 5. than N; and if equal, equal; and if leſs, leſs[d]. and G, K are any equimultiples whatever of A, D; and L, N any whatever of C, F; as therefore A is to C, ſo is D to F.



Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which, taken two and two in a croſs order, have the ſame ratio, viz. A to B, as G to H; B to C, as F to G; and C to D, as E to F. A is to D, as E to H.

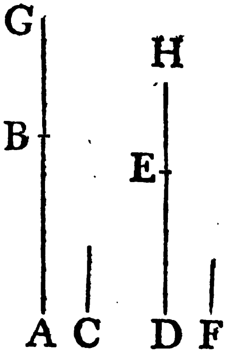
A.B. C.D.
E.F. G.H.

Becauſe A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a croſs order, have the ſame ratio; by the firſt caſe, A is to C, as F to H. but C is to D, as E is to F; wherefore again, by the firſt caſe, A is to D, as E to H. and ſo on, whatever be the number of magnitudes. Therefore if there be any number, &c. Q. E. D.

PROP. XXIV. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth; and the fifth to the ſecond the ſame ratio which the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond, the ſame ratio which the third and ſixth together have to the fourth. See N.

Let AB the firſt have to C the ſecond the ſame ratio, which DE the third has to F the fourth; and let BG the fifth have to C the ſecond the ſame ratio, which EH the ſixth has to F the fourth. AG, the firſt and fifth together, ſhall have to C the ſecond the ſame ratio, which DH, the third and ſixth together, has to F the fourth.



Becauſe BG is to C, as EH to F; by inverſion C is to BG, as F to EH. and becauſe as AB is to C, ſo is DE to F; and as C to BG, ſo F to EH; ex aequali [a] AB is to BG, as DE to EH. and becauſe theſe magnitudes are proportionals, they ſhall likewiſe be proportionals when taken jointly [b]; as therefore AG is to GB, ſo is DH to HE; but as GB to C, ſo is HE to F. Therefore, ex aequali [a], as AG is to C, ſo is DH to F. Wherefore if the firſt, &c. Q. E. D.

a. 22. 5.
b. 18. 5.

COR. 1. If the ſame Hypotheſis be made as in the Propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond, as the exceſs of the third and ſixth to the fourth. the Demonſtration of this is the ſame with that of the Propoſition, if Diviſion be uſed inſtead of Compoſition.

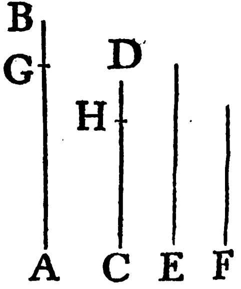
Cor. 2. The Propoſition holds true of two ranks of magnitudes, whatever be their number, of which each of the firſt rank has to a ſecond magnitude the ſame ratio that the correſponding one of the ſecond rank has to a fourth magnitude; as is manifeſt.

PROP. XXV. THEOR.

IF four magnitudes are proportionals, the greateſt and leaſt of them together are greater than the other two together.

Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greateſt of them, and a. A, & 14. 5. conſequently F the leaſt [a]. AB together with F are greater than CD together with E.

Take AG equal to E, and CH equal to F. then becauſe as AB to CD, ſo is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG to CH. and becauſe AB the whole is to the whole CD, as AG is to CH; likewiſe the remainder GB ſhall be to the remainder HD, as b. 19. 5. the whole AB is to the whole [b] CD. but c. A. 5. AB is greater than CD, therefore [c] GB is greater than HD. and becauſe AG is equal to E, and CH to F; AG and F together are equal to CH and E together. If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD; AB and F together are greater than CD and E. Therefore if four magnitudes, &c. Q. E. D.



PROP. F. THEOR.

See N. RATIOS which are compounded of the ſame ratios, are the ſame with one another.

Let A be to B, as D to E; and B to C, as E to F. the ratio which is compounded of the ratios of A to B, and B to C, which, by the Definition of compound ratio, is the ratio of A to C, is the ſame with the ratio of D to F, which, by the ſame Definition, is compounded of the ratios of D to E, and E to F.

A. B. C.
D. E. F.

Becauſe there are three magnitudes A, B, C, and three others D, E, F which taken two and two in order have the ſame ratio; ex aequali, A is to C, as D to F[a]. a. 22. 5.

Next, Let A be to B, as E to F; and B to C, as D to E; therefore, *ex aequali, in proportione perturbata*[b], A is to C, as D to F; that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the ſame with the ratio of D to F, which is compounded of the ratios of D to E, and E to F. and in like manner the Propoſition may be demonſtrated whatever be the number of ratios in either caſe. b. 23. 5.

A. B. C.
D. E. F.

PROP. G. THEOR.

IF ſeveral ratios be the ſame with ſeveral ratios, each to each; the ratio which is compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame with the other ratios, each to each. See N.

Let A be to B, as E to F; and C to D, as G to H. and let A be to B, as K to L; and C to D, as L to M. then the ratio of K to M, by the Definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the ſame with the ratios of A to B, and C to D. and as E to F, ſo let N be to O; and as G to H, ſo let O be to P; then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the ſame with the ratios of E to F, and G to H. and it is to be ſhewn that the ratio of K to M, is the ſame with the ratio of N to P, or that K is to M, as N to P.

A. B. C. D. K. L. M.
E. F. G. H. N. O. P.

Becauſe K is to L, as (A to B, that is, as E to F, that is as) N to O; and as L to M, ſo is (C to D, and ſo is G to H, and ſo is) O to P. ex aequali [a], K is to M, as N to P. Therefore if ſeveral ratios, &c. Q. E. D.

a. 22. 5.

PROP. H. THEOR.

See N.

IF a ratio compounded of ſeveral ratios be the ſame with a ratio compounded of any other ratios, and if one of the firſt ratios, or a ratio compounded of any of the firſt, be the ſame with one of the laſt ratios, or with the ratio compounded of any of the laſt; then the ratio compounded of the remaining ratios of the firſt, or the remaining ratio of the firſt, if but one remain, is the ſame with the ratio compounded of thoſe remaining of the laſt, or with the remaining ratio of the laſt.

Let the firſt ratios be thoſe of A to B, B to C, C to D, D to E, and E to F; and let the other ratios be thoſe of G to H, H to K, K to L, and L to M. alſo let the ratio of A to F, which is compounded [a] of the firſt ratios be the ſame with the ratio of G to M, which is compounded of the other ratios. and beſides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the ſame with the ratio of G to K, which is compounded of the ratios of G to H, and H to K. then the ratio compounded of the remaining firſt ratios, to wit, of the ratios of D to E, and E to F, which compounded ratio is the ratio of D to F, is the ſame with the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

a. Definition of compounded ratio.

A. B. C. D. E. F.
G. H. K. L. M.

Becauſe, by the Hypotheſis, A is to D, as G to K, by inverſion [b], D is to A, as K to G; and as A is to F, ſo is G to M; therefore [c], ex aequali, D is to F, as K to M. If therefore a ratio which is, &c. Q. E. D.

b. B. 5.
c. 22. 5.

PROP. K. THEOR.

See N.

IF there be any number of ratios, and any number of other ratios ſuch, that the ratio compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame, each to each, with the laſt ratios; and if one of the firſt ratios, or the ratio which is compounded of ratios which are the ſame with ſeveral of the firſt ratios, each to each, be the ſame with one of the laſt ratios, or with the ratio compounded of ratios which are the ſame, each to each, with ſeveral of the laſt ratios. then the ratio compounded of ratios which are the ſame with the remaining ratios of the firſt, each to each, or the remaining ratio of the firſt, if but one remain; is the ſame with the ratio compounded of ratios which are the ſame with thoſe remaining of the laſt, each to each, or with the remaining ratio of the laſt.

Let the ratios of A to B, C to D, E to F be the firſt ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios. and let A be to B, as S to T; and C to D, as T to V; and E to F, as V to X. therefore, by the Definition of compound ratio, the ratio of S to X is compounded of the ratios of S to T, T to V, and V to X, which are the ſame with the ratios of A to B, C to D, E to F, each to each. alſo as G to H, ſo let Y be to Z, and K to L, as Z to a; M to N, as a to b; O to P, as b to c; and Q to R, as c to d. therefore, by the ſame Definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

c, and c to d, which are the ſame, each to each, with the ratios of G to H, K to L, M to N, O to P, and Q to R. therefore, by the Hypotheſis, S is to X, as Y to d. alſo let the ratio of A to B, that is, the ratio of S to T, which is one of the firſt ratios, be the ſame with the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the Hypotheſis, are the ſame with the ratios of G to H, and K to L, two of the other ratios; and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the ſame with the remaining firſt ratios, viz. of C to D, and E to F; alſo let the ratio of m to p be that which is compounded of the ratios of m to n, n to o, and o to p, which are the ſame, each to each, with the remaining other ratios, viz. of M to N, O to P, and Q to R. then the ratio of h to l is the ſame with the ratio of m to p, or h is to l, as m to p.

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

Becauſe e is to f, as (G to H, that is as) Y to Z; and f is to g, as (K to L, that is as) Z to a; therefore, ex aequali, e is to g, as Y to a. and, by the Hypotheſis, A is to B, that is S to T, as e to g; wherefore S is to T, as Y to a, and, by inverſion, T is to S, as a to Y; and S is to X, as Y to d; therefore, ex aequali, T is to X, as a to d. alſo becauſe h is to k, as (C to D, that is as) T to V; and k is to l, as (E to F, that is as) V to X; therefore, ex aequali, h is to l, as T to X. in like manner it may be demonſtrated that m is to p, as a to d. and it was ſhewn that T is to X, as a to d. therefore [a] h is to l, as m to p. Q. E. D.

a. 11. 5.

The Propoſitions G and K are uſually, for the ſake of brevity, expreſſed in the ſame terms with Propoſitions F and H. and therefore it was proper to ſhew the true meaning of them when they are ſo expreſſed; eſpecially ſince they are very frequently made uſe of by Geometers.

THE

ELEMENTS

OF

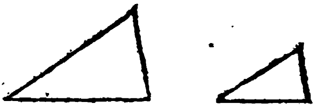
EUCLID.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are thoſe which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportionals.



II.

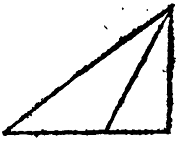
" Reciprocal figures, viz. triangles and parallelograms, are ſuch as " have their ſides about two of their angles proportionals in " ſuch manner, that a ſide of the firſt figure is to a ſide of the " other as the remaining ſide of this other is to the remaining " ſide of the firſt." See N.

III.

A ſtraight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

The altitude of any figure is the ſtraight line drawn from its vertex perpendicular to the baſe.



PROP. I. THEOR.

See N.

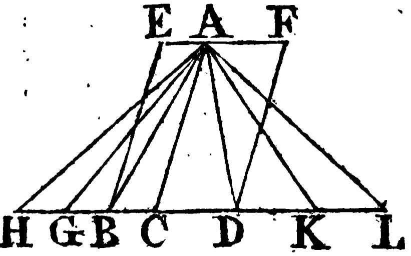
TRIANGLES and parallelograms of the ſame altitude are one to another as their baſes.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the ſame altitude, viz. the perpendicular drawn from the point A to BD. then as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of ſtraight lines BG, GH, each equal to the baſe BC; and DK, KL, any number of them, each equal to the baſe CD; and join AG, AH, AK, AL. then becauſe CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all equal [a]. therefore whatever multiple the baſe HC is of the baſe BC, the ſame multiple is the triangle AHC of the triangle ABC. for the ſame reaſon whatever multiple the baſe LC is of the baſe CD, the ſame multiple is the triangle ALC of the triangle ADC. and if the baſe HC be equal to the baſe CL, the triangle AHC is alſo equal to the triangle ALC [a]; and if the baſe HC be greater than the baſe CL, likewiſe the triangle AHC is greater than the triangle ALC; and if leſs, leſs. therefore ſince there are four magnitudes, viz. the two baſes BC, CD, and the two triangles ABC, ACD; and of the baſe BC and the triangle ABC the firſt and third, any equimultiples whatever have been taken, viz. the baſe HC and triangle AHC; and of the baſe CD and triangle ACD the ſecond and fourth have been taken any equimultiples whatever, viz. the baſe CL and triangle ALC; and that it has been ſhewn that if the baſe HC be greater than the baſe CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if leſs, leſs. Therefore [b] as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD.

a. 38. 1.

b. 5. Def. 5.



And becauſe the parallelogram CE is double of the triangle
ABC [c], and the parallelogram CF double of the triangle ACD,
and that magnitudes have the ſame ratio which their equimultiples c. 41. 1.
have [d]; as the triangle ABC is to the triangle ACD, ſo is the parallelogram d. 15. 5.
EC to the parallelogram CF. and becauſe it has been
ſhewn that as the baſe BC is to the baſe CD, ſo is the triangle ABC
to the triangle ACD; and as the triangle ABC to the triangle
ACD, ſo is the parallelogram EC to the parallelogram CF; therefore
as the baſe BC is to the baſe CD, ſo is [e] the parallelogram EC e. 11. 5.
to the parallelogram CF. Wherefore triangles, &c. Q. E. D.

Cor. From this it is plain that triangles and parallelograms that have equal altitudes, are one to another as their baſes.

Let the figures be placed ſo as to have their baſes in the ſame
ſtraight line; and having drawn perpendiculars from the vertices of
the triangles to the baſes, the ſtraight line which joins the vertices
is parallel to that in which their baſes are [f], becauſe the perpendiculars f. 33. 1.
are both equal and parallel to one another. then, if the
ſame conſtruction be made as in the Propoſition, the Demonſtration
will be the ſame.

PROP. II. THEOR.

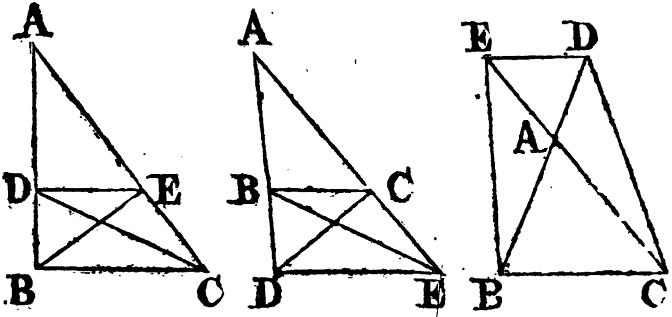
If a ſtraight line be drawn parallel to one of the ſides of a triangle, it ſhall cut the other ſides, or theſe produced, proportionally. and if the ſides, or the ſides produced be cut proportionally, the ſtraight line which joins the points of ſection ſhall be parallel to the remaining ſide of the triangle. See N.

Let DE be drawn parallel to BC one of the ſides of the triangle ABC. BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle
CDE [a], becauſe they are on the ſame baſe DE, and between the a. 37. 1.
ſame parallels DE, BC. ADE is another triangle, and equal magnitudes
have to the ſame, the ſame ratio [b]; therefore as the triangle b. 7. 5.
BDE to the triangle ADE, ſo is the triangle CDE to the triangle
ADE; but as the triangle BDE to the triangle ADE, ſo is [c] BD to c. 1. 6.
DA, becauſe having the ſame altitude, viz. the perpendicular drawn
from the point E to AB, they are to one another as their baſes. and

for the ſame reaſon, as the triangle CDE to the triangle ADE, ſo is CE to EA. Therefore as BD to DA; ſo is CE to EA[d].

d. 11. 5. Next, Let the ſides AB, AC of the triangle ABC, or theſe produced, be cut proportionally in the points D, E, that is, ſo that BD be to DA, as CE to EA; and join DE. DE is parallel to BC.



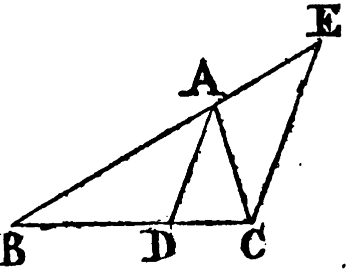
The ſame conſtruction being made, becauſe as BD to DA, ſo
is CE to EA; and as BD to DA, ſo is the triangle BDE to the
c. 1. 6. triangle ADE[c]; and as CE to EA, ſo is the triangle CDE to the
triangle ADE; therefore the triangle BDE is to the triangle ADE,
as the triangle CDE to the triangle ADE, that is, the triangles
BDE, CDE have the ſame ratio to the triangle ADE; and there-
e. 9. 5. fore[e] the triangle BDE is equal to the triangle CDE. and they are
on the ſame baſe DE; but equal triangles on the ſame baſe are
f. 39. 1. between the ſame parallels[f]; therefore DE is parallel to BC.
Wherefore if a ſtraight line, &c. Q. E. D.

PROP. III. THEOR.

IF the angle of a triangle be divided into two equal angles, by a ſtraight line which alſo cuts the baſe; the ſegments of the baſe ſhall have the ſame ratio which the other ſides of the triangle have to one another. and if the ſegments of the baſe have the ſame ratio which the other ſides of the triangle have to one another, the ſtraight line drawn from the vertex to the point of ſection divides the vertical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided in two equal angles by the ſtraight line AD. BD is to DC, as BA to AC,

Thro' the point C draw CE parallel [a] to DA, and let BA pro-
duced meet CE in E. Becauſe the ſtraight line AC meets the pa-
rallels AD, EC, the angle ACE is equal to the alternate angle a. 31. 1.
CAD [b]. but CAD, by the Hypotheſis, is equal to the angle BAD; b. 29. 1.
wherefore BAD is equal to the angle ACE. Again, becauſe the
ſtraight line BAE meets the parallels AD, EC, the outward angle BAD is equal to the inward and oppoſite angle AEC. but the angle ACE has been proved equal to the angle BAD; therefore alſo ACE is equal to the angle AEC, and conſequently the
ſide AE is equal to the ſide [c] AC. c. 6. 1.
and becauſe AD is drawn parallel to one of the ſides of the tri-
angle BCE, viz. to EC, BD is to DC, as BA to AE [d]; but AE is d. 2. 6.
equal to AC; therefore as BD to DC, ſo is BA to AC [e]. e. 7. 5.



Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the ſtraight line AD.

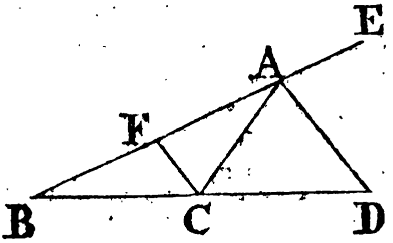
The ſame conſtruction being made, becauſe as BD to DC, ſo
is BA to AC; and as BD to DC, ſo is BA to AE [d], becauſe AD
is parallel to EC; therefore BA is to AC, as BA to AE [f]. con- f. 11. 5.
ſequently AC is equal to AE [g], and the angle AEC is therefore g. 9. 5.
equal to the angle ACE [h]. but the angle AEC is equal to the out- h. 5. 1.
ward and oppoſite angle BAD; and the angle ACE is equal to
the alternate angle CAD [b]. wherefore alſo the angle BAD is e-
qual to the angle CAD. therefore the angle BAC is cut into two
equal angles by the ſtraight line AD. Therefore if the angle,
&c. Q. E. D.

PROP. A. THEOR.

IF the outward angle of a triangle made by producing one of its ſides, be divided into two equal angles, by a ſtraight line which alſo cuts the baſe produced; the ſegments between the dividing line and the extremities of the baſe have the ſame ratio which the other ſides of the triangle have to one another. and if the ſegments of the baſe produced have the ſame ratio which the other ſides of the triangle have, the ſtraight line drawn from the vertex to the point of ſection divides the outward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the ſtraight line AD which meets the baſe produced in D. BD is to DC, as BA to AC.

a. 31. 1. Through C draw CF parallel to AD[a]; and becauſe the ſtraight
line AC meets the parallels AD, FC, the angle ACF is equal to
b. 29. 1. the alternate angle CAD[b]. but CAD is equal to the angle DAE[c];
c. Hyp. therefore alſo DAE is equal to the angle ACF. Again, becauſe
the ſtraight line FAE meets the parallels AD, FC, the outward
angle DAE is equal to the inward and oppoſite angle CFA. but the angle ACF has been proved equal to the angle DAE; therefore alſo the angle ACF is equal to the angle CFA, and conſequently the ſide AF is equal to the ſide
d. 6. 1. AC[d]. and becauſe AD is parallel to FC a ſide of the triangle BCF,
e. 2. 6. BD is to DC, as BA to AF[e]; but AF is equal to AC; as there-
fore BD is to DC, ſo is BA to AC.



Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

The ſame conſtruction being made, becauſe BD is to DC, as
BA to AC; and that BD is alſo to DC, as BA to AF[e]; therefore
f. 11. 5. BA is to AC, as BA to AF[f]. wherefore AC is equal to AF[g],
g. 9. 5.
h. 5. 1. and the angle AFC equal[h] to the angle ACF. but the angle AFC

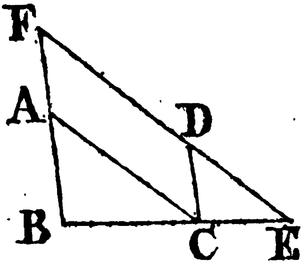
is equal to the outward angle EAD, and the angle ACF to the alternate angle CAD; therefore alſo EAD is equal to the angle CAD. Wherefore if the outward, &c. Q. E. D.

PROP. IV. THEOR.

THE ſides about the equal angles of equiangular triangles are proportionals; and thoſe which are oppoſite to the equal angles are homologous ſides, that is, are the antecedents or conſequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC
equal to the angle DCE, and the angle ACB to the angle DEC,
and conſequently [a] the angle BAC equal to the angle CDE. The a. 32. 1.
ſides about the equal angles of the triangles ABC, DCE are proportionals; and thoſe are the homologous ſides which are oppoſite to the equal angles.

Let the triangle DCE be placed ſo that its ſide CE may be contiguous to BC, and in the ſame ſtraight line with it. and becauſe
the angles ABC, ACB are together leſs than two right angles [b]; b. 17. 1.
ABC and DEC, which is equal to ACB, are alſo leſs than two right angles. wherefore BA, ED produced
ſhall meet [c]; let them be produced c. 12. Ax. 1.
and meet in the point F. and becauſe the angle ABC is equal to the angle
DCE, BF is parallel [d] to CD. Again, d. 28. 1.
becauſe the angle ACB is equal to the angle DEC, AC is parallel to FE [d]. therefore FACD is a parallelogram; and conſequently AF is
equal to CD, and AC to FD [e]. and becauſe AC is parallel to FE e. 34. 1.
one of the ſides of the triangle FBE, BA is to AF, as BC to CE [f]. f. 2. 6.
but AF is equal to CD, therefore [g] as BA to CD, ſo is BC to CE; g. 7. 5.
and alternately, as AB to BC, ſo DC to CE. Again, becauſe CD is parallel to BF, as BC to CE, ſo is FD to DE [f]; but FD is equal to AC; therefore as BC to CE, ſo is AC to DE. and alternately, as BC to CA, ſo CE to ED. therefore becauſe it has been proved that AB is to BC, as DC to CE; and as BC to CA, ſo CE to ED,
ex aequali [h], BA is to AC, as CD to DE. Therefore the ſides, &c. h. 22. 5.
Q. E. D.

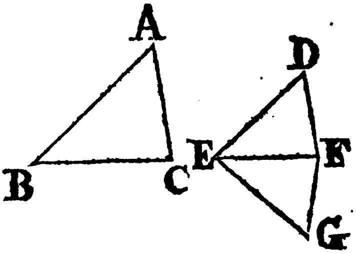


PROP. V. THEOR.

IF the ſides of two triangles, about each of their angles, be proportionals, the triangles ſhall be equiangular, and have their equal angles oppoſite to the homologous ſides.

Let the triangles ABC, DEF have their ſides proportionals, ſo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and conſequently, ex aequali, BA to AC, as ED to DF. the triangle ABC is equiangular to the triangle DEF, and their equal angles are oppoſite to the homologous ſides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and alſo BAC to EDF.

a. 23. 1. At the points E, F in the ſtraight line EF make [a] the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle b. 32. 1. EGF [b], and the triangle ABC is therefore equiangular to the triangle GEF; and conſequently they have their ſides oppoſite to c. 4. 6. the equal angles proportionals [c]. wherefore as AB to BC, ſo is GE to EF; but as AB to BC, d. 11. 5. ſo is DE to EF; therefore as DE to EF, ſo [d] GE to EF. therefore DE and GE have the ſame ratio to EF, and conſequently are e. 9. 5. equal [e]. for the ſame reaſon DF is equal to FG. and becauſe, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two ſides DE, EF are equal to the two GE, EF, and the baſe DF f. 8. 1. is equal to the baſe GF; therefore the angle DEF is equal [f] to the angle GEF, and the other angles to the other angles which are g. 4. 1. ſubtended by the equal ſides [g]. Wherefore the angle DFE is equal to the angle GFE, and EDF to EGF. and becauſe the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF. for the ſame reaſon, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore if the ſides, &c. Q. E. D.

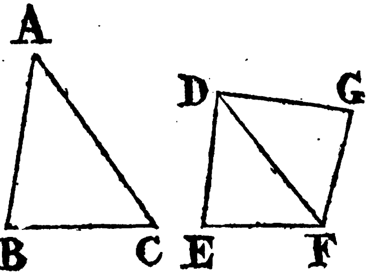


PROP. VI. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about the equal angles proportionals, the triangles ſhall be equiangular, and ſhall have thoſe angles equal which are oppoſite to the homologous ſides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the ſides about thoſe angles proportionals; that is, BA to AC, as ED to DF. The triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

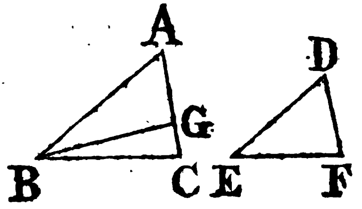
At the points D, F, in the ſtraight line DF, make [a] the angle a. 23. 1.
FDG equal to either of the angles BAC, EDF; and the angle DFG
equal to the angle ACB. wherefore the remaining angle at B is
equal to the remaining one at G [b], and conſequently the tri- b. 32. 1.
angle ABC is equiangular to the triangle DGF; and there-
fore as BA to AC, ſo is [c] GD to DF. but, by the Hypotheſis, c. 4. 6.
as BA to AC, ſo is ED to DF; as therefore ED to DF, ſo is [d] d. 11. 5.
GD to DF; wherefore ED is
equal [e] to DG; and DF is common to the two triangles EDF, GDF. e. 9. 5.
therefore the two ſides ED, DF are equal to the two ſides GD,
DF; and the angle EDF is equal to the angle GDF, wherefore
the baſe EF is equal to the baſe FG [f], and the triangle EDF to f. 4. 1.
the triangle GDF, and the remaining angles to the remaining angles,
each to each, which are ſubtended by the equal ſides. therefore
the angle DFG is equal to the angle DFE, and the angle at G to
the angle at E. but the angle DFG is equal to the angle ACB;
therefore the angle ACB is equal to the angle DFE. and the angle
BAC is equal to the angle EDF [g]; wherefore alſo the remaining g. Hyp.
angle at B is equal to the remaining angle at E. Therefore the
triangle ABC is equiangular to the triangle DEF. Wherefore if
two triangles, &c. Q. E. D.



PROP. VII. THEOR.

See N. **IF two triangles have one angle of the one equal to one angle of the other, and the ſides about two other angles proportionals; then if each of the remaining angles be either leſs, or not leſs than a right angle; or if one of them be a right angle: the triangles ſhall be equiangular, and have thoſe angles equal about which the ſides are proportionals.**

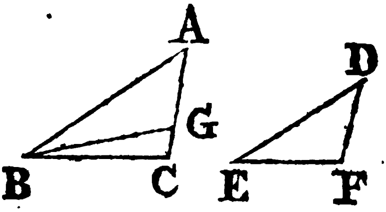
Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the ſides about two other angles ABC, DEF proportionals, ſo that AB is to BC, as DE to EF; and, in the firſt caſe, let each of the remaining angles at C, F be leſs than a right angle. The triangle ABC is equiangular to the triangle DEF, viz. the angle ABC is equal to the angle DEF, and the remaining angle at C, to the remaining angle at F.



For if the angles ABC, DEF be not equal, one of them is greater
than the other; let ABC be the greater, and at the point B in the
ſtraight line AB make the an-
a. 23. 1. gle ABG equal to the angle [a]
DEF. and becauſe the angle at
A is equal to the angle at D,
and the angle ABG to the an-
gle DEF; the remaining angle
b. 32. 1. AGB is equal [b] to the remain-
ing angle DFE. therefore the
c. 4. 6. triangle ABG is equiangular to the triangle DEF; wherefore [c] as
AB is to BG, ſo is DE to EF; but as DE to EF, ſo, by Hypothe-
d. 11. 5. ſis, is AB to BC; therefore as AB to BC, ſo is AB to BG [d]; and
becauſe AB has the ſame ratio to each of the lines BC, BG;
e. 9. 5. BC is equal [e] to BG, and therefore the angle BGC is equal to
f. 5. 1. the angle BCG [f]. but the angle BCG is, by Hypotheſis, leſs than
a right angle; therefore alſo the angle BGC is leſs than a right
angle, and the adjacent angle AGB muſt be greater than a right
g. 13. 1. angle [g]. But it was proved that the angle AGB is equal to the
angle at F; therefore the angle at F is greater than a right angle.
but, by the Hypotheſis, it is leſs than a right angle; which is ab-

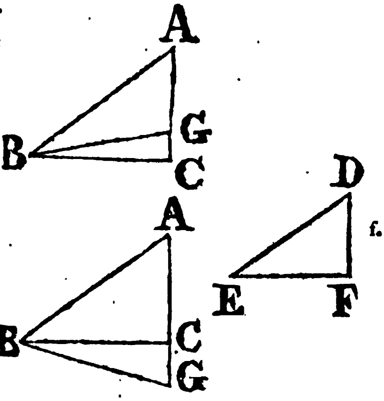
ſurd. Therefore the angles ABC, DEF are not unequal, that is, they are equal. and the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F. therefore the triangle ABC is equiangular to the triangle DEF.

Next, Let each of the angles at C, F be not leſs than a right angle. the triangle ABC is alſo in this caſe equiangular to the triangle DEF.



The ſame conſtruction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC. but the angle at C is not leſs than a right angle; therefore the angle BGC is not leſs than a right angle. wherefore two angles of the triangle BGC are together not leſs than two right angles; which is impoſſible[h]; and therefore the triangle ABC may be proved to h. 17. 1.
be equiangular to the triangle DEF, as in the firſt caſe.

Laſtly, Let one of the angles at C, F, viz. the angle at C be a right angle; in this caſe likewiſe the triangle ABC is equiangular to the triangle DEF.



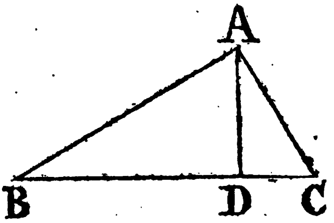
For if they be not equiangular, make at the point B of the ſtraight line AB the angle ABG equal to the angle DEF; then it may be proved, as in the firſt caſe, that BG is equal to BC. but the angle BCG is a right angle, therefore[f] the angle BGC f. 5. 1.
is alſo a right angle; whence two of the angles of the triangle BGC are together not leſs than two right angles; which is impoſſible[h]. therefore the triangle ABC is equiangular to the triangle DEF. Wherefore if two triangles, &c. Q. E. D.

PROP. VIII. THEOR.

See N. **IN a right angled triangle, if a perpendicular be drawn from the right angle to the baſe; the triangles on each ſide of it are ſimilar to the whole triangle, and to one another.**

Let ABC be a right angled triangle having the right angle BAC; and from the point A let AD be drawn perpendicular to the baſe BC. the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another.

Becauſe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to
a. 32. 1. the remaining angle BAD[a]. therefore the triangle ABC is equiangular to the triangle ABD, and the ſides about their equal angles
b. 4. 6. are proportionals[b], wherefore the
c. 1. Def. 6. triangles are ſimilar[c]. in the like manner it may be demonſtrated that the triangle ADC is ſimilar to the triangle ABC.



Alſo the triangles ABD, ADC are ſimilar to one another.

Becauſe the right angle BDA is equal to the right angle ADC, and alſo the angle BAD to the angle at C, as has been proved; the remaining angle at B is equal to the remaining angle DAC. therefore the triangle ABD is equiangular and ſimilar[c] to the triangle ADC. Therefore in a right angled, &c. Q. E. D.

Cor. From this it is manifeſt that the perpendicular drawn from the right angle of a right angled triangle to the baſe, is a mean proportional between the ſegments of the baſe. and alſo that each of the ſides is a mean proportional between the baſe, and its ſegment adjacent to that ſide. becauſe in the triangles BDA, ADC, BD is to DA, as DA to DC[b]; and in the triangles ABC, DBA, BC is to BA, as BA to BD[b]; and in the triangles ABC, ACD, BC is to CA, as CA to CD[b].

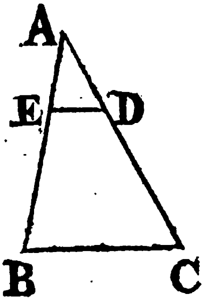
PROP. IX. PROB.

FROM a given ſtraight line to cut off any part required.

See N.

Let AB be the given ſtraight line; it is required to cut off any part from it.

From the point A draw a ſtraight line AC making any angle with AB; and in AC take any point D, and take AC which is the ſame multiple of AD that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it. then AE is the ſame part of AB that AD is of AC; that is, AE is the part required to be cut off.



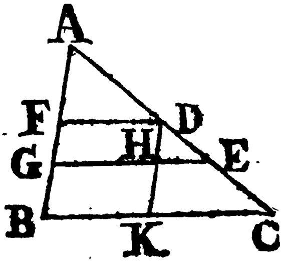
Becauſe ED is parallel to one of the ſides of the triangle ABC, viz. to BC, as CD is to DA, ſo is [a] BE to EA; and, by compoſition [b], CA is to AD, as BA to AE. but CA is a multiple of AD, therefore [c] BA is the ſame multiple of AE. whatever part therefore AD is of AC, AE is the ſame part of AB. wherefore from the ſtraight line AB the part required is cut off. Which was to be done.

a. 2. 6.
b. 18. 5.
c. D. 5.

PROP. X. PROB.

TO divide a given ſtraight line ſimilarly to a given divided ſtraight line, that is, into parts that ſhall have the ſame ratios to one another which the parts of the divided given ſtraight line have.

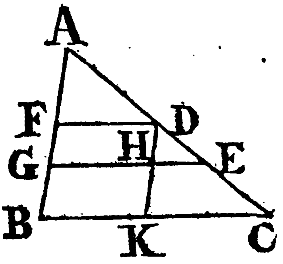
Let AB be the ſtraight line given to be divided, and AC the divided line; it is required to divide AB ſimilarly to AC.



Let AC be divided in the points D, E; and let AB, AC be placed ſo as to contain any angle, and join BC, and through the points D, E draw [a] DF, EG parallels to it; and through D draw DHK parallel to AB. therefore each of the figures FH, HB is a parallelogram;

a. 31. 1.

b. 34. 1. c. 2. 6. wherefore DH is equal [b] to FG, and HK to GB. and becauſe HE is parallel to KC one of the ſides of the triangle DKC, as CE to ED, ſo is [c] KH to HD. but KH is equal to BG, and HD to GF; therefore as CE to ED, ſo is BG to GF. again, becauſe FD is parallel to EG one of the ſides of the triangle AGE, as ED to DA, ſo is GF to FA. but it has been proved that CE is to ED, as BG to GF; as therefore CE to ED, ſo is BG to GF; and as ED to DA, ſo GF to FA. therefore the given ſtraight line AB is divided ſimilarly to AC. Which was to be done.



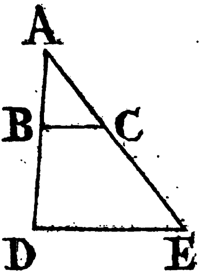
PROP. XI. PROB.

TO find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines, and let them be placed ſo as to contain any angle; it is required to find a third proportional to AB, AC.

a. 31. 1. Produce AB, AC to the points D, E; and make BD equal to AC, and having joined BC, thro' D draw DE parallel to it [a].

b. 2. 6. Becauſe BC is parallel to DE a ſide of the triangle ADE, AB is [b] to BD, as AC to CE. but BD is equal to AC, as therefore AB to AC, ſo is AC to CE. Wherefore to the two given ſtraight lines AB, AC a third proportional CE is found. Which was to be done.

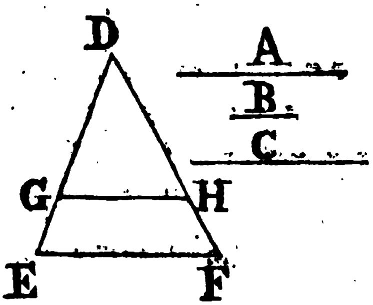


PROP. XII. PROB.

TO find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is required to find a fourth proportional to A, B, C.

Take two ſtraight lines DE, DF containing any angle EDF; and upon theſe make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel[a] to it through the point E. and becauſe GH is parallel to EF one of the ſides of the triangle DEF, DG is to GE, as DH to HF[b]. but DG is equal to A, GE to B, and DH to C; therefore as A is to B, ſo is C to HF. Wherefore to the three given ſtraight lines A, B, C a fourth proportional HF is found. Which was to be done.



a. 31. 1.

b. 2. 6.

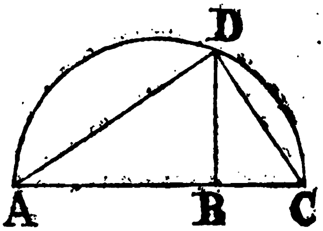
PROP. XIII. PROB.

TO find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required to find a mean proportional between them.

Place AB, BC in a ſtraight line, and upon AC deſcribe the ſemicircle ADC, and from the point B draw[a] BD at right angles to AC, and join AD, DC.

Becauſe the angle ADC in a ſemicircle is a right angle[b], and becauſe in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the baſe, DB is a mean proportional between AB, BC the ſegments of the baſe[c]. therefore between the two given ſtraight lines AB, BC, a mean proportional DB is found. Which was to be done.



a. 11. 1.

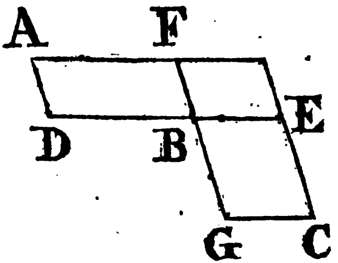
b. 31. 3.

c. Cor. 8. 6.

PROP. XIV. THEOR.

EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional. and parallelograms that have one angle of the one equal to one angle of the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms which have the angles at B equal, and let the ſides DB, BE be placed in the ſame ſtraight line; wherefore alſo FB, BG are in one ſtraight line [a]. the ſides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF. a. 14. 1.



Complete the parallelogram FE; and becauſe the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE [b]. but as AB to FE, ſo is the baſe DB to BE [c]; and as BC to FE, ſo is the baſe GB to BF; therefore as DB to BE, ſo is GB to BF [d]. Wherefore the ſides of the parallelograms AB, BC about their equal angles are reciprocally proportional. b. 7. 5. c. 1. 6. d. 11. 5.

But let the ſides about the equal angles be reciprocally proportional, viz. as DB to BE, ſo GB to BF; the parallelogram AB is equal to the parallelogram BC.

Becauſe as DB is to BE, ſo GB to BF; and as DB to BE, ſo is the parallelogram AB to the parallelogram FE; and as GB to BF, ſo is parallelogram BC to parallelogram FE; therefore as AB to FE, ſo BC to FE [d]. wherefore the parallelogram AB is equal [e] to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D. e. 9. 5.

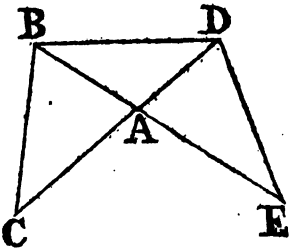
PROP. XV. THEOR.

EQUAL triangles which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional. and triangles which have one angle in the one equal to one angle in the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles which have the angle BAC equal to the angle DAE; the ſides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed ſo that their ſides CA, AD be in one ſtraight line; wherefore alſo EA and AB are in one ſtraight line [a]; and join BD. Becauſe the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD ſo is triangle EAD to triangle DAB [b]. but as triangle CAB to triangle BAD, ſo is the baſe CA to AD [c]; and as triangle EAD to triangle DAB, ſo is the baſe EA to AB [c]; as therefore CA to AD, ſo is EA to AB [d]. wherefore the ſides of the triangles ABC, ADE about the equal angles are reciprocally proportional.

a. 14. 1.
b. 7. 5.
c. 1. 6.
d. 11. 5.



But let the ſides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

Having joined BD as before, becauſe as CA to AD, ſo is EA to AB; and as CA to AD, ſo is triangle BAC to triangle BAD [c]; and as EA to AB, ſo triangle EAD to triangle BAD [c]; therefore [d] as triangle BAC to triangle BAD, ſo is triangle EAD to triangle BAD; that is, the triangles BAC, EAD have the ſame ratio to the triangle BAD. wherefore the triangle ABC is equal [e] to the triangle ADE. Therefore equal triangles, &c. Q. E. D.

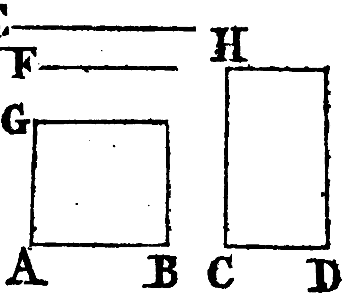
e. 9. 5.

PROP. XVI. THEOR.

IF four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means. and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines are proportionals.

Let the four ſtraight lines AB, CD, E, F be proportionals, viz. as AB to CD, ſo E to F; the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

a. 11. 1. From the points A, C draw [a] AG, CH at right angles to AB, CD; and make AG equal to F, and CH equal to E, and complete the parallelograms BG, DH. becauſe as AB to CD, ſo is E to F;
b. 7. 5. and that E is equal to CH, and F to AG; AB is [b] to CD, as CH to AG. therefore the ſides of the parallelograms BG, DH about the equal angles are reciprocally proportional; but parallelograms which have their ſides about equal angles reciprocally proportional,
c. 14. 6. are equal to one another[c]; therefore the parallelogram BG is equal to the parallelogram DH. and the parallelogram BG is contained by the ſtraight lines AB, F, becauſe AG is equal to F; and the parallelogram DH is contained by CD and E, becauſe CH is equal to E. therefore the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the ſtraight lines AB, F be equal to that which is contained by CD, E; theſe four lines are proportionals, viz. AB is to CD, as E to F.

The ſame conſtruction being made, becauſe the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, becauſe AG is equal to F; and the rectangle DH by CD, E, becauſe CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular. but the ſides about the equal angles of equal parallelograms are reciprocally proportional [c]. wherefore as AB to CD, ſo is CH to AG; and CH is equal to E, and AG to F. as therefore AB is to CD, ſo E to F. Wherefore if four, &c. Q. E. D.

c. 14. 6.

PROP. XVII. THEOR.

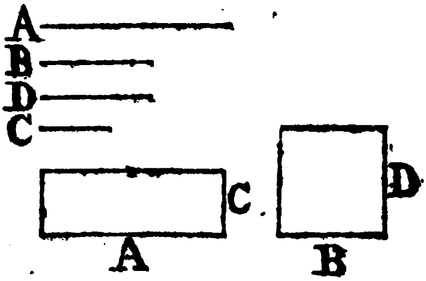
IF three ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the ſquare of the mean: and if the rectangle contained by the extremes be equal to the ſquare of the mean, the three ſtraight lines are proportionals.

Let the three ſtraight lines A, B, C be proportionals, viz. as A to B, ſo B to C; the rectangle contained by A, C is equal to the ſquare of B.

Take D equal to B; and becauſe as A to B, ſo B to C, and that B is equal to D; A is [a] to B, as D to C. but if four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means [b], therefore the rectangle contained by A, C is equal to that contained by B, D. but the rectangle contained by B, D is the ſquare of B; becauſe B is equal to D. therefore the rectangle contained by A, C is equal to the ſquare of B.

a. 7. 5.

b. 16. 6.



And if the rectangle contained by A, C be equal to the ſquare of B; A is to B, as B to C.

The ſame conſtruction being made, becauſe the rectangle contained by A, C is equal to the ſquare of B, and the ſquare of B is equal to the rectangle contained by B, D, becauſe B is equal to D; therefore the rectangle contained by A, C is equal to that contained by B, D. but if the rectangle contained by the extremes be equal to that contained by the means, the four ſtraight lines are proportionals [b]. therefore A is to B, as D to C; but B is equal to D;

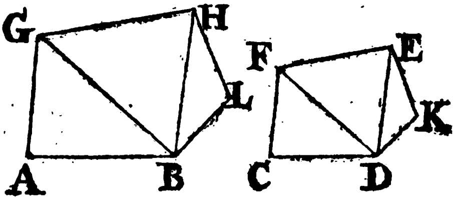
wherefore as A to B, ſo B to C. Therefore if three ſtraight lines, &c. Q. E. D.

PROP. XVIII. PROB.

UPON a given ſtraight line to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to a given rectilineal figure.

Let AB be the given ſtraight line, and CDEF the given rectilineal figure of four ſides; it is required upon the given ſtraight line AB to deſcribe a rectilineal figure ſimilar and ſimilarly ſituated to CDEF.

a. 23. 1. Join DF, and at the points A, B in the ſtraight line AB make [a] the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefore the remaining angle CFD is equal to the b. 32. 1. remaining angle AGB [b]. wherefore the triangle FCD is equiangular to the triangle GAB. again, at the points G, B in the ſtraight line GB make [a] the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH. then becauſe the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE. for the ſame reaſon, the angle ABH is equal to the angle CDE; alſo the angle at A is equal to the angle at C, and the angle GHB to FED. therefore the rectilineal figure ABHG is equiangular to CDEF. but likewiſe theſe figures have their ſides about the equal angles proportionals. becauſe the triangles GAB, c. 4. 6. FCD being equiangular, BA is [c] to AG, as DC to CF; and becauſe AG is to GB, as CF to FD; and as GB to GH, ſo, by reaſon of the equiangular triangles BGH, DFE, is FD to FE; d. 22. 5. therefore, ex aequali [d], AG is to GH, as CF to FE. in the ſame manner it may be proved that AB is to BH, as CD to DE. and GH is to HB, as FE to ED [c]. Wherefore becauſe the rectilineal



figures ABHG, CDEF are equiangular, and have their ſides about
the equal angles proportionals, they are ſimilar to one another[c].

Next, Let it be required to deſcribe upon a given ſtraight line c. 1. Def. 6.
AB, a rectilineal figure ſimilar, and ſimilarly ſituated to the recti-
lineal figure CDKEF of five ſides.

Join DE, and upon the given ſtraight line AB deſcribe the rectilineal figure ABHG ſimilar and ſimilarly ſituated to the quadrilateral figure CDEF, by the former caſe. and at the points B, H in the ſtraight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L. and becauſe the figures ABHG, CDEF are ſimilar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK. for the ſame reaſon, the angle ABL is equal to the angle CDK. therefore the five ſided figures AGHLB, CFEKD are equiangular. and
becauſe the figures AGHB, CFED are ſimilar, GH is to HB, as
FE to ED; and as HB to HL, ſo is ED to EK[c]; therefore ex c. 4. 6.
æquali[d], GH is to HL, as FE to EK. for the ſame reaſon, AB is d. 22. 5.
to BL, as CD to DK. and BL is to LH, as[c] DK to KE, becauſe
the triangles BLH, DKE are equiangular. therefore becauſe the five ſided figures AGHLB, CFEKD are equiangular, and have their ſides about the equal angles proportionals, they are ſimilar to one another. and in the ſame manner a rectilineal figure of ſix ſides may be deſcribed upon a given ſtraight line ſimilar to one given, and ſo on. Which was to be done.

PROP. XIX. THEOR.

SIMILAR triangles are to one another in the duplicate ratio of their homologous ſides.

Let ABC, DEF be ſimilar triangles having the angle B equal to
the angle E, and let AB be to BC, as DE to EF, ſo that the ſide
BC is homologous to EF[a]. the triangle ABC has to the triangle a. 12. Def. 5.
DEF, the duplicate ratio of that which BC has to EF.

Take BG a third proportional to BC, EF[b], ſo that BC is to EF, b. 11. 6.
as EF to BG, and join GA. then, becauſe as AB to BC, ſo DE to
EF; alternately[c], AB is to DE, as BC to EF. but as BC to EF, ſo c. 16. 5.

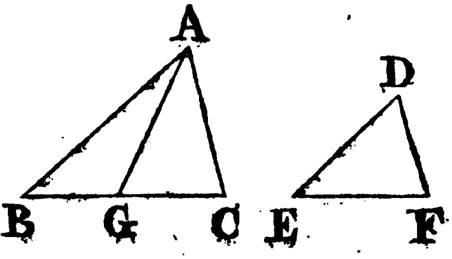
is EF to BG; therefore [d] as AB to DE, ſo is EF to BG. where-fore the ſides of the triangles ABG, DEF which are about the e-

d. 11. 5. qual angles are reciprocally proportional. but triangles which have the ſides about two equal angles reciprocally proportional are equal

e. 15. 6. to one another [e]. therefore the triangle ABG is equal to the triangle DEF. and becauſe as BC is to EF, ſo EF to BG; and that if three ſtraight lines be propor-

f. 10. Def. 5. tionals, the firſt is ſaid [f] to have to the third the duplicate ratio of that which it has to the ſecond; BC therefore has to BG the duplicate ratio of that which BC has to EF. but as

g. 1. 6. BC to BG, ſo is [g] the triangle ABC to the triangle ABG. therefore the triangle ABC has to the triangle ABG, the duplicate ratio of that which BC has to EF. but the triangle ABG is equal to the triangle DEF; wherefore alſo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. therefore ſimilar triangles, &c. Q. E. D.



Cor. From this it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any triangle upon the firſt to a ſimilar and ſimilarly deſcribed triangle upon the ſecond.

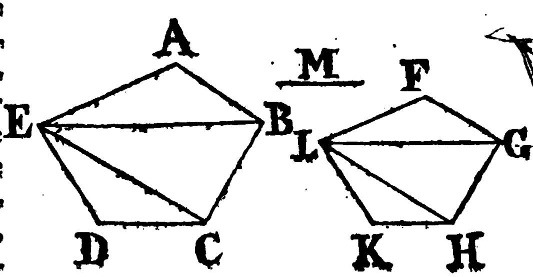
PROP. XX. THEOR.

SIMILAR polygons may be divided into the ſame number of ſimilar triangles, having the ſame ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous ſides have.

Let ABCDE, FGHKL be ſimilar polygons, and let AB be the homologous ſide to FG. the polygons ABCDE, FGHKL may be divided into the ſame number of ſimilar triangles, whereof each to each has the ſame ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the ſide FG.

Join BE, EC, GL, LH. and becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE is equal to the angle GFL [a], and BA is to AE, as GF to FL [a]. wherefore becauſe the triangles ABE, FGL have one angle in one equal to an angle in the other, and their ſides about theſe equal angles proportionals, the triangle ABE is equiangular [b], and therefore ſimilar to the triangle FGL [c]; wherefore the angle ABE is equal to the angle FGL. and, becauſe the polygons are ſimilar, the whole angle ABC is equal [a] to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH. and becauſe the triangles ABE, FGL are ſimilar, EB is to BA, as LG to GF [a]; and alſo, becauſe the polygons are ſimilar, AB is to BC, as FG to GH [a]; therefore, ex aequali [d], EB is to BC, as LG to GH; that is, the ſides about the equal angles EBC, LGH are proportionals; therefore [b] the triangle EBC is equiangular to the triangle LGH, and ſimilar to it [c]. for the ſame reaſon the triangle ECD likewiſe is ſimilar to the triangle LHK. therefore the ſimilar polygons ABCDE, FGHKL are divided into the ſame number of ſimilar triangles.

a. 1. Def. 6.
b. 6. 6.
c. 4. 6.
d. 22. 5.

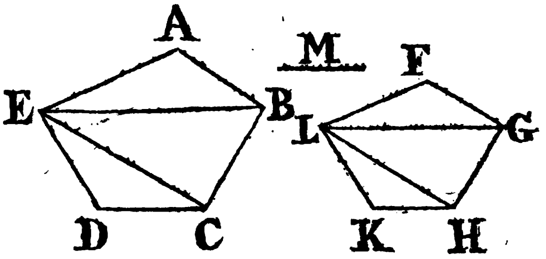


Alſo theſe triangles have, each to each, the ſame ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the conſequents FGL, LGH, LHK. and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the homologous ſide FG.

Becauſe the triangle ABE is ſimilar to the triangle FGL, ABE has to FGL the duplicate ratio [e] of that which the ſide BE has to the ſide GL. for the ſame reaſon, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL. therefore as the triangle ABE to the triangle FGL, ſo [f] is the triangle BEC to the triangle GLH. Again, becauſe the triangle EBC is ſimilar to the triangle LGH, EBC has to LGH, the duplicate ratio of that which the ſide EC has to the ſide LH. for the ſame reaſon, the triangle ECD has to the triangle LHK, the duplicate ratio of that which EC has to LH. as therefore the triangle EBC to the triangle LGH, ſo is [f] the

e. 19. 6.
f. 11. 5.

triangle ECD to the triangle LHK. but it has been proved that the triangle EBC is likewiſe to the triangle LGH, as the triangle ABE to the triangle FGL. Therefore as the triangle ABE is to the triangle FGL, ſo is triangle EBC to triangle LGH, and triangle ECD to triangle LHK. and therefore as one of the antecedents to one of the conſequents, ſo are all the antecedents to all the
g. 12. 5. conſequents [g]. Wherefore as the triangle ABE to the triangle FGL, ſo is the polygon ABCDE to the polygon FGHKL. but the triangle ABE has to the triangle FGL, the duplicate ratio of that which the ſide AB has to the homologous ſide FG. Therefore alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous ſide FG. Wherefore ſimilar polygons, &c. Q. E. D.



Cor. 1. In like manner it may be proved that ſimilar four ſided figures, or of any number of ſides are one to another in the duplicate ratio of their homologous ſides, and it has already been proved in triangles. Therefore univerſally, ſimilar rectilineal figures are to one another in the duplicate ratio of their homologous ſides.

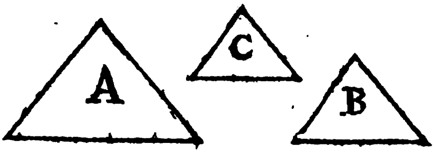
Cor. 2. And if to AB, FG two of the homologous ſides a
h. 10. Def. 5. third proportional M be taken, AB has [h] to M the duplicate ratio of that which AB has to FG. but the four ſided figure or polygon upon AB has to the four ſided figure or polygon upon FG likewiſe the duplicate ratio of that which AB has to FG. therefore as AB is to M, ſo is the figure upon AB to the figure upon FG,
i. Cor. 19. 6. which was alſo proved in triangles [i]. Therefore, univerſally, it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any rectilineal figure upon the firſt, to a ſimilar and ſimilarly deſcribed rectilineal figure upon the ſecond.

PROP. XXI. THEOR.

RECTILINEAL figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another.

Let each of the rectilineal figures A, B be ſimilar to the rectilineal figure C. the figure A is ſimilar to the figure B.

Becauſe A is ſimilar to C, they are equiangular, and alſo have
their ſides about the equal angles proportionals [a]. Again, becauſe a. 1. Def. 6.
B is ſimilar to C, they are equiangular, and have their ſides about the equal angles proportionals [a]. therefore the figures A, B are each of
them equiangular to C, and have the ſides about the equal angles
of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular [b], and have their ſides about b. 1. Ax. 1.
the equal angles proportionals [c]. Therefore A is ſimilar [a] to B. c. 11. 5.
Q. E. D.



PROP. XXII. THEOR.

IF four ſtraight lines be proportionals, the ſimilar rectilineal figures ſimilarly deſcribed upon them ſhall alſo be proportionals. and if the ſimilar rectilineal figures ſimilarly deſcribed upon four ſtraight lines be proportionals, thoſe ſtraight lines ſhall be proportionals.

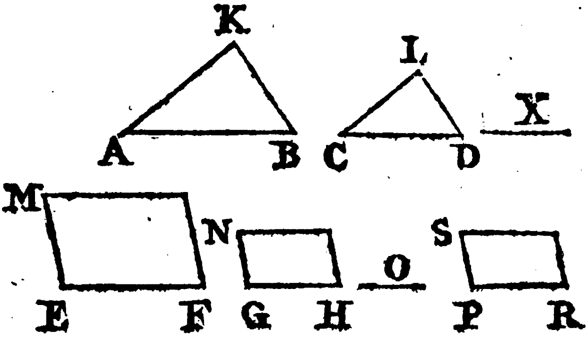
Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD let the ſimilar rectilineal figures KAB, LCD be ſimilarly deſcribed; and upon EF, GH the ſimilar rectilineal figures MF, NH, in like manner. the rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional [a] X; and to EF, GH a a. 11. 6.
third proportional O. and becauſe AB is to CD, as EF to GH, b. 11. 5.
therefore CD is [b] to X, as GH to O; wherefore ex aequali [c], as AB c. 22. 5.

to X, ſo EF to O. but as AB to X, ſo is [d] the rectilineal KAB to the rectilineal LCD, and as EF to O, ſo is [d] the rectilineal MF to
d. 2. Cor. 20. 6. the rectilineal NH. therefore as KAB to LCD, ſo [b] is MF to NH.

b. 11. 5. And if the rectilineal KAB be to LCD, as MF to NH; the ſtraight line AB is to CD, as EF to GH.

e. 12. 6. f. 18. 6. Make [e] as AB to CD, ſo EF to PR, and upon PR deſcribe [f] the rectilineal figure SR ſimilar and ſimilarly ſituated to either of



the figures MF, NH. then becauſe as AB to CD, ſo EF to PR, and that upon AB, CD are deſcribed the ſimilar and ſimilarly ſituated rectilineals KAB, LCD, and upon EF, PR, in like manner, the ſimilar rectilineals MF, SR; KAB is to LCD, as MF to SR; but, by the Hypotheſis, KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the ſame ratio to each of the two NH,
g. 9. 5. SR, theſe are equal [g] to one another. they are alſo ſimilar, and ſimilarly ſituated; therefore GH is equal to PR. and becauſe as AB to CD, ſo is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four ſtraight lines, &c. Q. E. D.

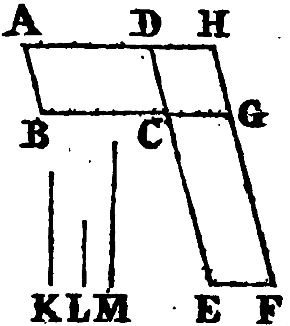
PROP. XXIII. THEOR.

See N. EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratios of their ſides.

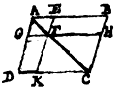
Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG. the ratio of the parallelogram AC to the parallelogram CF, is the ſame with the ratio which is compounded of the ratios of their ſides.

Let BC, CG be placed in a ſtraight line, therefore DC and CE are alſo in a ſtraight line [a]; and complete the parallelogram DG, and, taking any ſtraight line K, make [b] as BC to CG, ſo K to L; and as DC to CE, ſo make [b] L to M. therefore the ratios of K to L, and L to M are the ſame with the ratios of the ſides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is ſaid to be compounded [c] of the ratios of K to L, and L to M: wherefore alſo K has to M, the ratio compounded of the ratios of the ſides. and becauſe as BC to CG, ſo is the parallelogram AC to the parallelogram CH [d]; but as BC to CG, ſo is K to L; therefore K is [e] to L, as the parallelogram AC to the parallelogram CH. again, becauſe as DC to CE, ſo is the parallelogram CH to the parallelogram CF; but as DC to CE, ſo is L to M; wherefore L is [e] to M, as the parallelogram CH to the parallelogram CF. therefore ſince it has been proved that as K to L, ſo is the parallelogram AC to the parallelogram CH; and as L to M, ſo the parallelogram CH to the parallelogram CF; ex aequali [f], K is to M, as the parallelogram AC to the parallelogram CF. but K has to M the ratio which is compounded of the ratios of the ſides; therefore alſo the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the ſides. Wherefore equiangular parallelograms, &c. Q. E. D.

a. 14. 1.
b 12. 6.
c. A. Def. 5.
d. 1. 6.
e. 11. 5.
f. 22. 5.



PROP. XXIV. THEOR.



THE parallelograms about the diameter of any parallelogram, are ſimilar to the whole, and to one another.

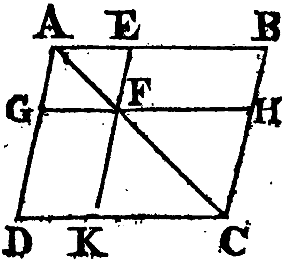
See N.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter. the parallelograms EG, HK are ſimilar both to the whole parallelogram ABCD, and to one another.

Becauſe DC, GF are parallels, the angle ADC is equal [a] to the angle AGF. for the ſame reaſon, becauſe BC, EF are parallels, the

a. 29. 1.

angle ABC is equal to the angle AEF. and each of the angles
BCD, EFG is equal to the oppoſite angle DAB [b], and therefore are
b. 34. 1. equal to one another; wherefore the parallelograms ABCD, AEFG
are equiangular. and becauſe the angle ABC is equal to the angle
AEF, and the angle BAC common to the two triangles BAC, EAF,
they are equiangular to one another;
c. 4. 6. therefore [c] as AB to BC, ſo is AE to
EF. and becauſe the oppoſite ſides of
parallelograms are equal to one another [b], AB is [d] to AD, as AE to AG;
d. 7. 5. and DC to CB, as GF to FE; and alſo
CD to DA, as FG to GA. therefore
the ſides of the parallelograms ABCD,
AEFG about the equal angles are pro-
e. 1.Def. 6. portionals; and they are therefore ſimilar to one another [e]. for the
ſame reaſon, the parallelogram ABCD is ſimilar to the parallelogram FHCK. Wherefore each of the parallelograms GE, KH
is ſimilar to DB. but rectilineal figures which are ſimilar to the
f. 21. 6. ſame rectilineal figure, are alſo ſimilar to one another [f], therefore
the parallelogram GE is ſimilar to KH. Wherefore the parallelograms, &c. Q. E. D.



PROP. XXV. PROB.

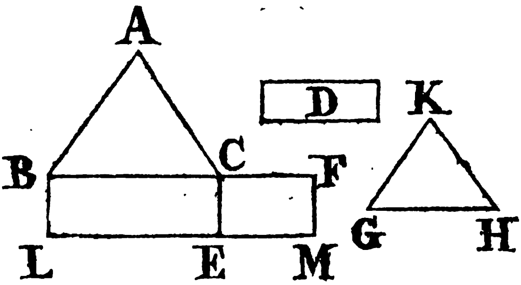
See N. TO deſcribe a rectilineal figure which ſhall be ſimilar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be deſcribed is required to be ſimilar, and D that to which it muſt be equal. It is required to deſcribe a rectilineal figure ſimilar to ABC and equal to D.

a. Cor. 45. 1. Upon the ſtraight line BC deſcribe [a] the parallelogram BE equal
to the figure ABC; alſo upon CE deſcribe [a] the parallelogram CM
equal to D, and having the angle FCE equal to the angle CBL.
b. 29. 1. 14. 1. therefore BC and CF are in a ſtraight line [b], as alſo LE and EM.
c. 13. 6. between BC and CF find [c] a mean proportional GH, and upon
d. 18. 6. GH deſcribe [d] the rectilineal figure KGH ſimilar and ſimilarly
ſituated to the figure ABC. and becauſe BC is to GH, as GH to
e. 2. Cor. 20. CF, and if three ſtraight lines be proportionals, as the firſt is to
6. the third, ſo is [e] the figure upon the firſt to the ſimilar and ſimilarly

Book VI.

deſcribed figure upon the ſecond; therefore as BC to CF, ſo is the rectilineal figure ABC to KGH. but as BC to CF, ſo is [f] the parallelogram BE to the parallelogram EF. f. 1. 6. therefore as the rectilineal figure ABC is to KGH, ſo is the parallelogram BE to the parallelogram EF [g]. and the rectilineal figure ABC is equal to the pa- g. 11. 5.

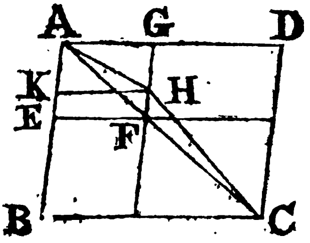


rallelogram BE; therefore the rectilineal figure KGH is equal [h] h. 14. 5. to the parallelogram EF. but EF is equal to the figure D, wherefore alſo KGH is equal to D; and it is ſimilar to ABC. Therefore the rectilineal figure KGH has been deſcribed ſimilar to the figure ABC, and equal to D. Which was to be done.

PROP. XXVI. THEOR.

IF two ſimilar parallelograms have a common angle, and be ſimilarly ſituated; they are about the ſame diameter.

Let the parallelograms ABCD, AEFG be ſimilar and ſimilarly ſituated, and have the angle DAB common. ABCD and AEFG are about the ſame diameter.



For if not, let, if poſſible, the parallelogram BD have its diameter AHC in a different ſtraight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and thro' H draw HK parallel to AD or BC. therefore the parallelograms ABCD, AKHG being about the ſame diameter, they are ſimilar to one another [a]. where- a. 24. 6. fore as DA to AB, ſo is [b] GA to AK. but becauſe ABCD and b. 1. Def. 6. AEFG are ſimilar parallelograms, as DA is to AB ſo is GA to AE.

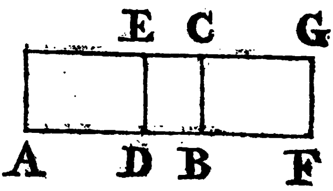
c. 11. 5.
d. 9. 5.

therefore [c] as GA to AE, ſo GA to AK; wherefore GA has the ſame ratio to each of the ſtraight lines AE, AK; and conſequently AK is equal [d] to AE, the leſs to the greater, which is impoſſible. therefore ABCD and AKHG are not about the ſame diameter; wherefore ABCD and AEFG muſt be about the ſame diameter. Therefore if two ſimilar, &c. Q. E. D.

‘To underſtand the three following propoſitions more eaſily, it is to be obſerved,

‘1. That a parallelogram is ſaid to be applied to a ſtraight line, when it is deſcribed upon it as one of its ſides. Ex. gr. the parallelogram AC is ſaid to be applied to the ſtraight line AB.

‘2. But a parallelogram AE is ſaid to be applied to a ſtraight line AB, deficient by a parallelogram, when AD the baſe of AE is leſs than AB, and therefore AE is leſs than the parallelogram AC deſcribed upon AB in the ſame angle, and between the ſame parallels, by the parallelogram DC; and DC is therefore called the defect of AE.



‘3. And a parallelogram AG is ſaid to be applied to a ſtraight line AB, exceeding by a parallelogram, when AF the baſe of AG is greater than AB, and therefore AG exceeds AC the parallelogram deſcribed upon AB in the ſame angle, and between the ſame parallels, by the parallelogram BG.’

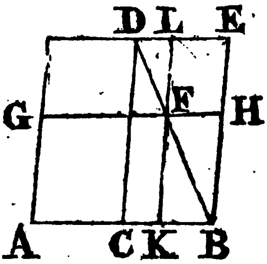
PROP. XXVII. THEOR.

See N.

OF all parallelograms applied to the ſame ſtraight line, and deficient by parallelograms ſimilar and ſimilarly ſituated to that which is deſcribed upon the half of the line; that which is applied to the half, and is ſimilar to its defect, is the greateſt.

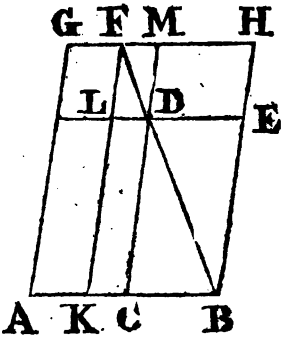
Let AB be a ſtraight line divided into two equal parts in C; and let the parallelogram AD be applied to the half AC, which is therefore deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB. of all the parallelograms applied to any other parts of AB and deficient by parallelograms that are ſimilar and ſimilarly ſituated to CE, AD is the greateſt.

Let AF be any parallelogram applied to AK any other part of AB than the half, ſo as to be deficient from the parallelogram upon the whole line AB by the parallelogram KH ſimilar and ſimilarly ſituated to CE; AD is greater than AF.



Firſt, Let AK the baſe of AF be greater than AC the half of AB; and becauſe CE is ſimilar to the parallelogram KH, they are about the ſame diameter [a]. draw their diameter DB, and complete the ſcheme. becauſe the parallelogram CF is equal [b] to FE, add KH to both, therefore the whole CH is equal to the whole KE. but CH is equal [c] to CG, becauſe the baſe AC is equal to the baſe CB; therefore CG is equal to KE. to each of theſe add CF; then the whole AF is equal to the gnomon CHL. therefore CE or the parallelogram AD is greater than the parallelogram AF.

a. 26. 6.
b. 43. 1.
c. 36. 1.



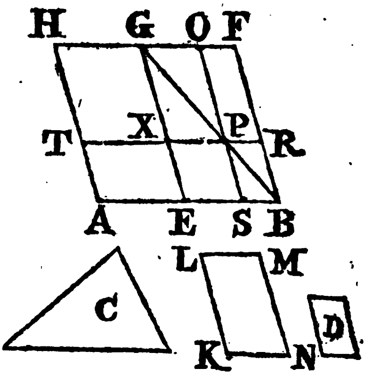
Next, Let AK the baſe of AF be leſs than AC, and, the ſame conſtruction being made, the parallelogram DH is equal to DG [c], for HM is equal to MG [d], becauſe BC is equal to CA; wherefore DH is greater than LG. but DH is equal [b] to DK; therefore DK is greater than LG. to each of theſe add AL; then the whole AD is greater than the whole AF. Therefore of all parallelograms applied, &c. Q. E. D.

d. 34. 1.

PROP. XXVIII. PROB.

See N. TO a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram ſimilar to a given parallelogram. but the given rectilineal figure to which the parallelogram to be applied is to be equal, muſt not be greater than the parallelogram applied to half of the given line having its defect ſimilar to the defect of that which is to be applied; that is, to the given parallelogram.

Let AB be the given ſtraight line, and C the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure muſt not be greater than the parallelogram applied to the half of the line having its defect from that upon the whole line ſimilar to the defect of that which is to be applied; and let D be the parallelogram to which this defect is required to be ſimilar. It is required to apply a parallelogram to the ſtraight line AB, which ſhall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram ſimilar to D.



Divide AB into two equal parts [a] in the point E, and upon EB deſcribe the parallelogram EBFG ſimilar [b] and ſimilarly ſituated to D, and complete the parallelogram AG, which muſt either be equal to C, or greater than it, by the determination. and if AG be equal to C, then what was required is already done; for upon the ſtraight line AB the parallelogram AG is applied equal to the figure C, and deficient by the parallelogram EF ſimilar to D. but if AG be not equal to C, it is greater than it; and EF is equal to AG, therefore EF alſo is greater than C. Make [c] the parallelogram KLMN equal to the exceſs of EF above C, and ſimilar and ſimilarly ſituated to D; but D is ſimilar to EF, therefore [d] alſo KM is

a. 10. 1.
b. 18. 6.
c. 25. 6.
d. 21. 6.

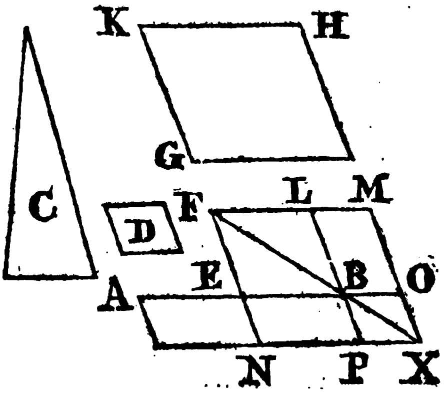
ſimilar to EF. let KL be the homologous ſide to EG, and LM to GF. and becauſe EF is equal to C and KM together, EF is greater than KM; therefore the ſtraight line EG is greater than KL, and GF than LM. make GX equal to LK, and GO equal to LM, and complete the parallelogram XGOP. therefore XO is equal and ſimilar to KM; but KM is ſimilar to EF; wherefore alſo XO is ſimilar to EF, and therefore XO and EF are about the ſame diameter [e]. let GPB be their diameter, and complete the ſcheme. then becauſe EF is equal to C and KM together, and XO a part of the one is equal to KM a part of the other, the remainder, viz. the gnomon ERO is equal to the remainder C. and becauſe OR is equal [f] to XS, by adding SR to each, the whole OB is equal to the whole XB. but XB is equal [g] to TE, becauſe the baſe AE is equal to the baſe EB; wherefore alſo TE is equal to OB. add XS to each, then the whole TS is equal to the whole, viz. to the gnomon ERO. but it has been proved that the gnomon ERO is equal to C, and therefore alſo TS is equal to C. Wherefore the parallelogram TS equal to the given rectilineal figure C, is applied to the given ſtraight line AB deficient by the parallelogram SR ſimilar to the given one D, becauſe SR is ſimilar to EF [h]. Which was to be done.

e. 26. 6.
f. 34. 1.
g. 36. 1.
h. 24. 6.

PROP. XXIX. PROB.

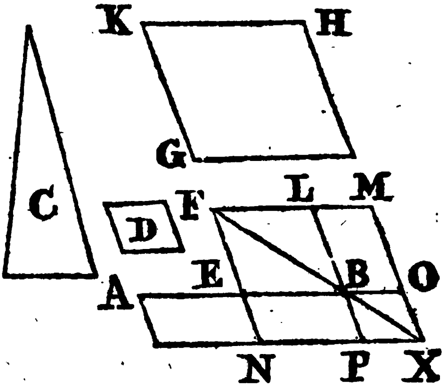
TO a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram ſimilar to another given. See N.

Let AB be the given ſtraight line, and C the given rectilineal figure to which the parallelogram to be applied is required to be equal, and D the parallelogram to which the exceſs of the one to be applied above that upon the given line is required to be ſimilar. It is required to apply



a parallelogram to the given ſtraight line AB which ſhall be equal to the figure C; exceeding by a parallelogram ſimilar to D.

Divide AB into two equal parts in the point E, and upon EB
a. 18. 6. deſcribe [a] the parallelogram EL ſimilar and ſimilarly ſituated to D.
b. 25. 6. and make [b] the parallelogram GH equal to EL and C together, and
c. 21. 6. ſimilar and ſimilarly ſituated to D; wherefore GH is ſimilar to EL [c].
let KH be the ſide homologous to FL, and KG to FE. and becauſe the parallelogram GH is greater than EL, therefore the ſide KH is greater than FL, and KG than FE. produce FL and FE, and make FLM equal to KH, and FEN to KG, and complete the parallelogram MN. MN is therefore equal and ſimilar to GH; but GH is ſimilar to EL; wherefore MN is ſimilar to EL, and conſequently EL and MN are about the ſame dia-
d. 26. 6. meter [d]. draw their diameter FX, and complete the ſcheme. therefore ſince GH is equal to EL and C together,



and that GH is equal to MN; MN is equal to EL and C. take away the common part EL; then the remainder, viz. the gnomon NOL is equal to C. and becauſe AE is equal to EB, the parallel-
e. 36. 1. ogram AN is equal [e] to the parallelogram NB, that is to BM [f].
f. 43. 1. add NO to each; therefore the whole, viz. the parallelogram AX is equal to the gnomon NOL. but the gnomon NOL is equal to C; therefore alſo AX is equal to C. Wherefore to the ſtraight line AB there is applied the parallelogram AX equal to the given rectilineal C, exceeding by the parallelogram PO, which is ſimilar
g. 24. 6. to D, becauſe PO is ſimilar to EL [g]. Which was to be done.

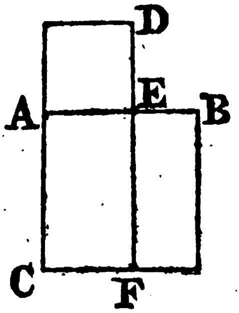
PROP. XXX. PROB.

To cut a given ſtraight line in extreme and mean ratio.

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Upon AB deſcribe[a] the ſquare BC, and to AC apply the parallelogram CD equal to BC exceeding by the figure AD ſimilar to BC[b]. but BC is a ſquare, therefore alſo AD is a ſquare. and becauſe BC is equal to CD, by taking the common part CE from each, the remainder BF is equal to the remainder AD. and theſe figures are equiangular, therefore their ſides about the equal angles are reciprocally proportional[c]. wherefore as FE to ED, ſo AE to EB. but FE is equal to AC[d], that is to AB; and ED is equal to AE. therefore as BA to AE, ſo is AE to EB. but AB is greater than AE; wherefore AE is greater than EB[e]. therefore the ſtraight line AB is cut in extreme and mean ratio in E[f]. Which was to be done.

a. 46. 1.
b. 29. 6.
c. 14. 6.
d. 34. 1.
e. 14. 5.
f. 3. Def. 6.

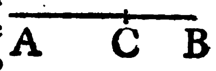


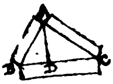
Otherwiſe,

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point C, ſo that the rectangle contained by AB, BC be equal to the ſquare of AC[g]. then becauſe the rectangle AB, BC is equal to the ſquare of AC, as BA to AC, ſo is AC to CB[h]. therefore AB is cut in extreme and mean ratio in C[f]. Which was to be done.

g. 11. 2.
h. 17. 6.





PROP. XXXI. THEOR.

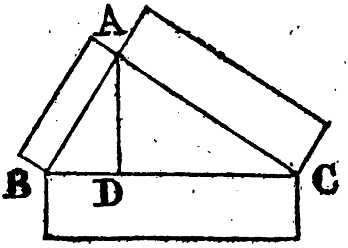
IN right angled triangles the rectilineal figure deſcribed upon the ſide oppoſite to the right angle, is equal to the ſimilar, and ſimilarly deſcribed figures upon the ſides containing the right angle. See N.

Let ABC be a right angled triangle, having the right angle BAC. the rectilineal figure deſcribed upon BC is equal to the ſimilar and ſimilarly deſcribed figures upon BA, AC.

Draw the perpendicular AD; therefore becauſe in the right angled triangle ABC, AD is drawn from the right angle at A perpendicular to the baſe BC, the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another[a]. and becauſe the triangle

a. 8. 6.

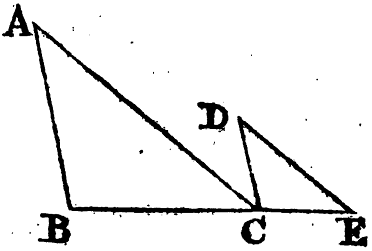
ABC is ſimilar to ABD, as CB to BA, ſo is BA to BD[b]. and becauſe theſe three ſtraight lines are proportionals, as the firſt to the b. 4. 6. third, ſo is the figure upon the firſt to the ſimilar, and ſimilarly deſcribed figure upon the c. 2. Cor. 20. 6. ſecond[c]. therefore as CB to BD, ſo is the figure upon CB to the ſimilar and ſimilarly deſcribed figure upon BA, d. B. 5. and, inverſely[d], as DB to BC, ſo is the figure upon BA to that upon BC. for the ſame reaſon, as DC to CB, ſo is the figure upon CA to that upon CB. Wherefore as BD and DC together to BC, ſo are the figures upon BA, AC to that upon e. 24. 5. BC[e]. but BD and DC together are equal to BC. Therefore the f. A. 5. figure deſcribed on BC is equal[f] to the ſimilar and ſimilarly deſcribed figures on BA, AC. Wherefore in right angled triangles, &c. Q. E. D.



PROP. XXXII. THEOR.

See N. IF two triangles which have two ſides of the one proportional to two ſides of the other, be joined at one angle ſo as to have their homologous ſides parallel to one another; the remaining ſides ſhall be in a ſtraight line.

Let ABC, DCE be two triangles which have the two ſides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE. BC and CE are in a ſtraight line.



Becauſe AB is parallel to DC, and the ſtraight line AC meets them, the alternate angles BAC, ACD are a. 29. 1. equal[a]; for the ſame reaſon the angle CDE is equal to the angle ACD; wherefore alſo BAC is equal to CDE. and becauſe the triangles ABC, DCE have one angle at A equal to one at D, and the ſides about theſe angles proportionals, viz. BA to AC, as CD to DE, the triangle ABC is equiangular[b] to DCE. therefore the angle ABC is equal to the angle DCE. and the angle BAC was proved to be equal to ACD. therefore the whole angle ACE is equal to the two angles ABC, BAC. add the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC, ACB. but ABC, BAC, ACB are equal to two right angles[c]; therefore alſo the angles ACE, ACB are equal to two right angles. and ſince at the point C in the ſtraight line AC, the two ſtraight lines BC, CE, which are on the oppoſite ſides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore[d] BC and CE are in a ſtraight line. Wherefore if two triangles, &c. Q. E. D.

b. 6. 6.
c. 32. 1.
d. 14. 1.

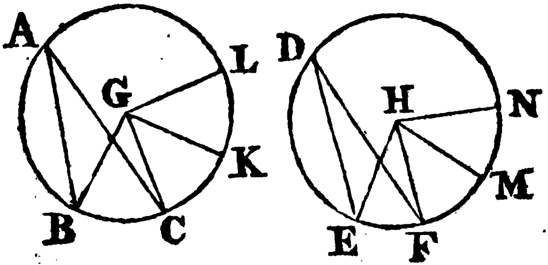
PROP. XXXIII. THEOR.

See N.

IN equal circles, angles whether at the centers or circumferences have the ſame ratio which the circumferences on which they ſtand have to one another. ſo alſo have the ſectors.

Let ABC, DEF be equal circles; and at their centers the angles BGC, EHF, and the angles BAC, EDF at their circumferences. as the circumference BC to the circumference EF, ſo is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and alſo the ſector BGC to the ſector EHF.

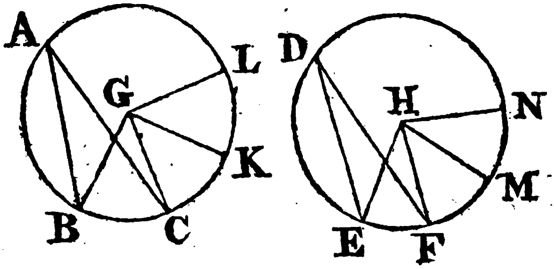
Take any number of circumferences CK, KL each equal to BC, and any number whatever FM, MN each equal to EF; and join GK, GL, HM, HN. Becauſe the circumferences BC, CK, KL are



all equal, the angles BGC, CGK, KGL are alſo all equal[a]. therefore what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the angle BGL of the angle BGC. for the ſame reaſon, whatever multiple the circumference EN is of

a. 27. 3.

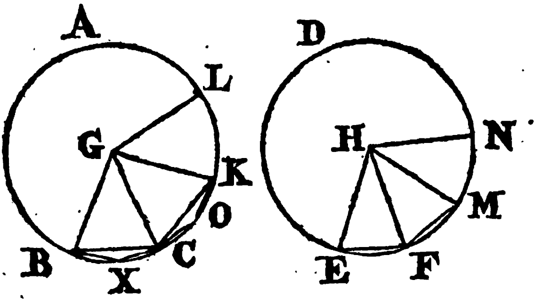
the circumference EF, the ſame multiple is the angle EHN of the
angle EHF. and if the circumference BL be equal to the circumfe-
a. 27. 3. rence EN, the angle BGL is alſo equal [a] to the angle EHN; and if
the circumference BL be greater than EN, likewiſe the angle BGL
is greater than EHN; and if leſs, leſs. there being then four magni-
tudes, the two circumferences BC, EF, and the two angles BGC,
EHF; of the circumference BC, and of the angle BGC, have been
taken any equimultiples whatever, viz. the circumference BL, and
the angle BGL; and of the circumference EF, and of the angle



EHF, any equimultiples whatever, viz. the circumference EN, and
the angle EHN. and it has been proved that if the circumference BL
be greater than EN, the angle BGL is greater than EHN; and if
equal, equal; and if leſs, leſs. as therefore the circumference BC
b. 5. Def. 5. to the circumference EF, ſo [b] is the angle BGC to the angle EHF.
c. 15. 5. but as the angle BGC is to the angle EHF, ſo is [c] the angle BAC to
d. 20. 3. the angle EDF, for each is double of each [d]. therefore as the cir-
cumference BC is to EF, ſo is the angle BGC to the angle EHF,
and the angle BAC to the angle EDF.

Alſo, as the circumference BC to EF, ſo is the ſector BGC to
the ſector EHF. Join BC, CK, and in the circumferences BC, CK
take any points X, O, and join BX, XC, CO, OK. then becauſe in
the triangles GBC, GCK the two ſides BG, GC are equal to the two
e. 4. 1. CG, GK, and that they contain equal angles; the baſe BC is equal [e]
to the baſe CK, and the triangle GBC to the triangle GCK. and
becauſe the circumference BC is equal to the circumference CK, the
remaining part of the whole circumference of the circle ABC, is
equal to the remaining part of the whole circumference of the ſame
circle. wherefore the angle BXC is equal to the angle COK [a]; and
f. 11. Def. 3. the ſegment BXC is therefore ſimilar to the ſegment COK [f]; and
they are upon equal ſtraight lines BC, CK. but ſimilar ſegments of
g. 24. 3. circles upon equal ſtraight lines, are equal [g] to one another. there-

fore the ſegment BXC is equal to the ſegment COK. and the triangle BGC is equal to the triangle CGK; therefore the whole, the ſector BGC is equal to the whole, the ſector CGK. for the ſame reaſon the ſector KGL is equal to each of the ſectors BGC, CGK. in the ſame manner the ſectors EHF, FHM, MHN may be proved equal to one another. therefore what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the ſector BGL of the ſector BGC. for the ſame reaſon, whatever multiple the circumference EN is of EF, the ſame multiple is the ſector EHN of the ſector EHF. and if the circumference BL be equal to EN, the ſector BGL is equal to the ſector EHN; and if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if leſs, leſs. ſince then there are four magnitudes, the two circumferences BC, EF, and the two ſectors BGC, EHF, and of the circumference BC and ſector BGC, the circumference BL and ſector BGL are any equimultiples whatever; and of the circumference EF and ſector EHF, the circumference EN and ſector EHN are any equimultiples whatever; and that it has been proved if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if equal, equal; and if leſs, leſs. Therefore [b] as the circumference BC is to the circumference EF, ſo is the ſector BGC to the ſector EHF. Wherefore in equal circles, &c. Q. E. D.



[b] 5. Def. 5.

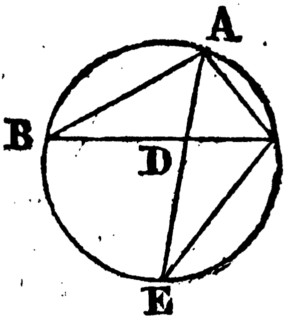
PROP. B. THEOR.

See N. IF an angle of a triangle be biſected by a ſtraight line, which likewiſe cuts the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the ſegments of the baſe, together with the ſquare of the ſtraight line biſecting the angle.

Let ABC be a triangle, and let the angle BAC be biſected by by the ſtraight line AD; the rectangle BA, AC is equal to the rectangle BD, DC together with the ſquare of AD.

a. 5. 4. Deſcribe the circle [a] ACB about the triangle, and produce AD to the circumference in E, and join EC. then becauſe the angle BAD is equal to the angle CAE, and the angle ABD to the angle [b] AEC, for they are in the ſame ſegment; the triangles ABD, AEC are equiangular to one another. therefore as BA to AD, ſo is [c] EA to AC, and conſequently the rectangle BA, AC is equal [d] to the rectangle EA, AD, that is [e] to the rectangle ED, DA together with the ſquare of AD. but the rectangle ED, DA is equal to the rectangle [f] BD, DC. Therefore the rectangle BA, AC is equal to the rectangle BD, DC together with the ſquare of AD. Wherefore if an angle, &c. Q. E. D.

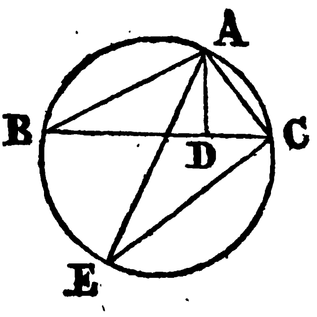
b. 21. 3.
c. 4. 6.
d. 16. 6.
e. 3. 2.
f. 35. 3.



PROP. C. THEOR.

See N. IF from an angle of a triangle a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the baſe BC; the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle deſcribed about the triangle.



Deſcribe [a] the circle ACB about the triangle, and draw its diameter AE, and join EC. becauſe the right angle BDA is equal [b] to the angle ECA in a ſemicircle, and the angle ABD to the angle AEC in the ſame ſegment [c]; the triangles ABD, AEC are equiangular. therefore as [d] BA to AD, ſo is EA to AC, and conſequently the rectangle BA, AC is equal [e] to the rectangle EA, AD. If therefore from an angle, &c. Q. E. D.

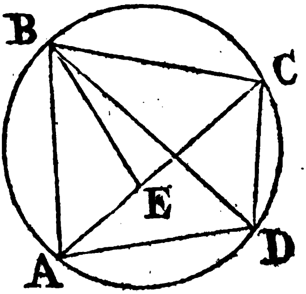
a. 5. 4.
b. 31. 3.
c. 21. 3.
d. 4. 6.
e. 16. 6.

PROP. D. THEOR.

THE rectangle contained by the diagonals of a quadrilateral inſcribed in a circle, is equal to both the rectangles contained by its oppoſite ſides.

Let ABCD be any quadrilateral inſcribed in a circle, and join AC, BD; the rectangle contained by AC, BD is equal to the two rectangles contained by AB, CD and by AD, BC *.

Make the angle ABE equal to the angle DBC; add to each of theſe the common angle EBD, then the angle ABD is equal to the angle EBC. and the angle BDA is equal [a] to the angle BCE, becauſe they are in the ſame ſegment; therefore the triangle ABD is equiangular to the triangle BCE. wherefore [b] as BC is to CE, ſo is BD to DA, and conſequently the rectangle BC, AD is equal [c] to the rectangle BD, CE. again, becauſe the angle ABE is equal to the angle DBC, and the angle [a] BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD. as therefore BA to AE, ſo is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE. but the rectangle BC, AD has been ſhewn equal to the rectangle BD, CE; therefore the whole rectangle AC, BD is equal to the rectangle AB, DC together with the rectangle AD, BC. Therefore the rectangle, &c. Q. E. D.



a. 21. 3.
b. 4. 6.
c. 16. 6.

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* This is a Lemma of Cl. Ptolemaeus in page 9. of his μεγάλη σύνταξις.

THE ELEMENTS OF EUCLID.

BOOK XI.

DEFINITIONS.

I.

A SOLID is that which hath length, breadth, and thickneſs.

II.

That which bounds a ſolid is a ſuperficies.

III.

A ſtraight line is perpendicular, or at right angles, to a plane, when it makes right angles with every ſtraight line meeting it in that plane.

IV.

A plane is perpendicular to a plane, when the ſtraight lines drawn in one of the planes perpendicularly to the common ſection of the two planes, are perpendicular to the other plane.

V.

The inclination of a ſtraight line to a plane is the acute angle contained by that ſtraight line, and another drawn from the point in which the firſt line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the firſt line above the plane, meets the ſame plane.

VI.

The inclination of a plane to a plane is the acute angle contained by two ſtraight lines drawn from any the ſame point of their common ſection at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are ſaid to have the ſame, or a like inclination to one another, which two other planes have, when the ſaid angles of inclination are equal to one another.

VIII.

Parallel planes are ſuch which do not meet one another tho' produced.

IX.

A ſolid angle is that which is made by the meeting of more than two plane angles, which are not in the ſame plane, in one point. See N.

X.

'The tenth Definition is omitted for reaſons given in the Notes.' See N.

XI.

Similar ſolid figures are ſuch as have all their ſolid angles equal, each to each, and which are contained by the ſame number of ſimilar planes. See N.

XII.

A Pyramid is a ſolid figure contained by planes that are conſtituted betwixt one plane and one point above it in which they meet.

XIII.

A Priſm is a ſolid figure contained by plane figures of which two that are oppoſite, are equal, ſimilar, and parallel to one another; and the others parallelograms.

XIV.

A Sphere is a ſolid figure deſcribed by the revolution of a ſemicircle about its diameter, which remains unmoved.

XV.

The axis of a ſphere is the fixed ſtraight line about which the ſemicircle revolves.

XVI.

The center of a ſphere is the ſame with that of the ſemicircle.

XVII.

The diameter of a ſphere is any ſtraight line which paſſes thro' the center, and is terminated both ways by the ſuperficies of the ſphere.

XVIII.

A Cone is a ſolid figure deſcribed by the revolution of a right angled triangle about one of the ſides containing the right angle, which ſide remains fixed.

If the fixed ſide be equal to the other ſide containing the right angle, the Cone is called a right angled Cone; if it be leſs than the other ſide, an obtuſe angled, and if greater, an acute angled Cone.

XIX.

The axis of a Cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a Cone is the circle deſcribed by that ſide containing the right angle, which revolves.

XXI.

A Cylinder is a ſolid figure deſcribed by the revolution of a right angled parallelogram about one of its ſides which remains fixed.

XXII.

The axis of a cylinder is the fixed ſtraight line about which the parallelogram revolves.

XXIII.

The baſes of a cylinder are the circles deſcribed by the two revolving oppoſite ſides of the parallelogram.

XXIV.

Similar cones and cylinders are thoſe which have their axis and the diameters of their baſes proportionals.

XXV.

A Cube is a ſolid figure contained by ſix equal ſquares.

XXVI.

A Tetrahedron is a ſolid figure contained by four equal and equilateral triangles.

XXVII.

An Octahedron is a ſolid figure contained by eight equal and equilateral triangles.

XXVIII.

A Dodecahedron is a ſolid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An Icoſahedron is a ſolid figure contained by twenty equal and equilateral triangles.

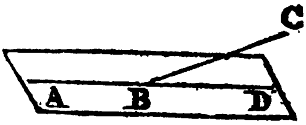
DEF. A.

A Parallelepiped is a ſolid figure contained by ſix quadrilateral figures whereof every oppoſite two are parallel.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane and See N. **another part above it.**

If it be poſſible, let AB part of the ſtraight line ABC be in the plane, and the part BC above it. and ſince the ſtraight line AB is in the plane, it can be produced in that plane. let it be produced to D. and let any plane paſs thro' the ſtraight line AD, and be turned about it until it paſs thro' the point C; and becauſe the points B, C are in this plane, the ſtraight line BC is in it [a]. therefore there are two ſtraight lines a. 7. Def. 1. ABC, ABD in the ſame plane that have a common ſegment AB, which is impoſſible [b]. Therefore one part, &c. Q. E. D. b. Cor. 11. 1.

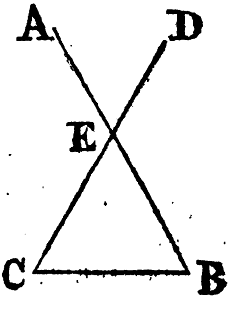


PROP. II. THEOR.

TWO ſtraight lines which cut one another are in See N. **one plane, and three ſtraight lines which meet one another are in one plane.**

Let two ſtraight lines AB, CD cut one another in E; AB, CD are in one plane. and three ſtraight lines EC, CB, BE which meet one another, are in one plane.

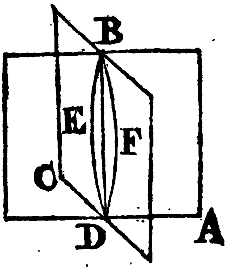
Let any plane paſs thro' the ſtraight line EB, and let the plane be turned about EB, produced if neceſſary, until it paſs thro' the point C. then becauſe the points E, C are in this plane, the ſtraight line EC is in it [a]. a. 7. Def. 1. for the ſame reaſon, the ſtraight line BC is in the ſame; and, by the Hypotheſis, EB is in it. therefore the three ſtraight lines EC, CB, BE are in one plane. but in the plane in which EC, EB are, in the ſame are [b] CD, b. 1. 11. AB. therefore AB, CD are in one plane. Wherefore two ſtraight lines, &c. Q. E. D.



PROP. III. THEOR.

See N. IF two planes cut one another, their common ſection is a ſtraight line.

Let two planes AB, BC cut one another, and let the line DB be their common ſection; DB is a ſtraight line. If it be not, from the point D to B draw in the plane AB the ſtraight line DEB, and in the plane BC the ſtraight line DFB. then two ſtraight lines DEB, DFB have the ſame extremities, and therefore include a ſpace betwixt them; which a. 10. Ax. 1. is impoſſible [a]. therefore BD the common ſection of the planes AB, BC cannot but be a ſtraight line. Wherefore if two planes, &c. Q. E. D.



PROP. IV. THEOR.

See N. IF a ſtraight line ſtand at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane which paſſes through them, that is, to the plane in which they are.

Let the ſtraight line EF ſtand at right angles to each of the ſtraight lines AB, CD in E the point of their interſection. EF is alſo at right angles to the plane paſſing thro' AB, CD.

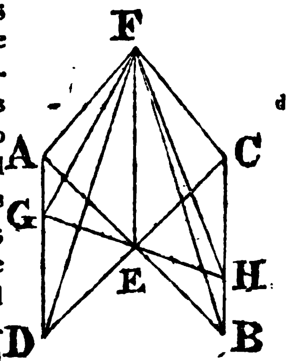
Take the ſtraight lines AE, EB, CE, ED all equal to one another; and thro' E draw, in the plane in which are AB, CD, any ſtraight line GEH; and join AD, CB; then from any point F in EF, draw FA, FG, FD, FC, FH, FB. and becauſe the two ſtraight lines AE, ED are equal to the two BE, EC, and that they contain a. 15. 1. equal angles [a] AED, BEC, the baſe AD is equal [b] to the baſe BC, b. 4. 1. and the angle DAE to the angle EBC. and the angle AEG is equal to the angle BEH [a]; therefore the triangles AEG, BEH have two angles of one equal to two angles of the other, each to each, and the ſides AE, EB, adjacent to the equal angles, equal to one another; c. 26. 1. wherefore they ſhall have their other ſides equal [c]. GE is therefore equal to EH, and AG to BH. and becauſe AE is equal to EB, and FE common and at right angles to them, the baſe AF is equal [b] to the baſe FB; for the ſame reaſon CF is equal to FD. and becauſe AD is equal to BC, and AF to FB, the two ſides FA, AD are equal to the two FB, BC, each to each; and the baſe DF was proved equal to the baſe FC; therefore the angle FAD is equal [d] to the angle FBC. again, it was proved that AG is equal to BH, and alſo AF to FB; FA then and AG, are equal to FB and BH, and the angle FAG has been proved equal to the angle FBH; therefore the baſe GF is equal [b] to the baſe FH. again, becauſe it was proved that GE is equal to EH, and EF is common; GE, EF are equal to HE, EF; and the baſe GF is equal to the baſe FH; therefore the angle GEF is equal [d] to the angle HEF, and conſequently each of theſe angles is a right [e] angle. Therefore FE makes right angles with GH, that is, with any ſtraight line drawn thro' E in the plane paſſing thro' AB, CD. In like manner it may be proved that FE makes right angles with every ſtraight line which meets it in that plane. But a ſtraight line is at right angles to a plane when it makes right angles with every ſtraight line which meets it in that plane [f]. therefore EF is at right angles to the plane in which are AB, CD. Wherefore if a ſtraight line, &c. Q. E. D.

b. 4. 1.

d. 8. 1.

e. 10. Def. 1.

f. 3. Def. 11.



PROP. V. THEOR.

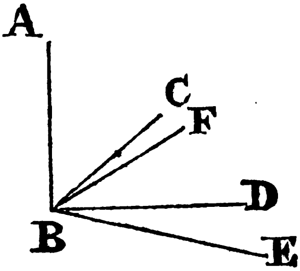
IF three ſtraight lines meet all in one point, and a ſtraight line ſtands at right angles to each of them in that point; theſe three ſtraight lines are in one and the ſame plane.

See N.

Let the ſtraight line AB ſtand at right angles to each of the ſtraight lines BC, BD, BE, in B the point where they meet; BC, BD, BE are in one and the ſame plane.

If not, let, if it be poſſible, BD and BE be in one plane, and BC be above it; and let a plane paſs through AB, BC, the common ſection of which with the plane, in which BD and BE are, ſhall be a ſtraight [a] line; let this be BF. therefore the three ſtraight lines AB, BC, BF are all in one plane, viz. that which paſſes thro' AB, BC. and becauſe AB ſtands at right angles to each of the ſtraight lines BD, BE, it is alſo at right angles [b] to the plane paſſing thro' them; and therefore makes right angles [c] with every ſtraight line meeting it in that plane; but BF which is in that plane meets it. therefore the angle ABF is a right angle; but the angle ABC, by the Hypotheſis, is alſo a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the ſame plane, which is impoſſible. therefore the ſtraight line BC is not above the plane in which are BD and BE. wherefore the three ſtraight lines BC, BD, BE are in one and the ſame plane. Therefore if three ſtraight lines, &c. Q. E. D.

a. 3. 11.
b. 4. 11.
c. 3. Def. 11.



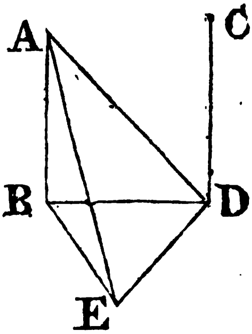
PROP. VI. THEOR.

IF two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel to one another.

Let the ſtraight lines AB, CD be at right angles to the ſame plane; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the ſtraight line BD, to which draw DE at right angles, in the ſame plane; and make DE equal to AB, and join BE, AE, AD. Then becauſe AB is perpendicular to the plane, it ſhall make right [a] angles with every ſtraight line which meets it, and is in that plane. but BD, BE, which are in that plane, do each of them meet AB. therefore each of the angles ABD, ABE is a right angle. for the ſame reaſon, each of the angles CDB, CDE is a right angle. and becauſe AB is equal to DE, and BD common, the two ſides AB, BD, are equal to the two ED, DB; and they contain right angles; therefore the baſe AD is equal [b] to the baſe BE. again, becauſe AB is equal to DE, and BE to AD; AB, BE are equal to ED,

a. 3. Def. 11.
b. 4. 1.



DA, and, in the triangles ABE, EDA, the baſe AE is common; therefore the angle ABE is equal [c] to the angle EDA. but ABE is a right angle; therefore EDA is alſo a right angle, and ED perpendicular to DA. but it is alſo perpendicular to each of the two BD, DC. wherefore ED is at right angles to each of the three ſtraight lines BD, DA, DC in the point in which they meet. therefore theſe three ſtraight lines are all in the ſame plane [d]. but AB is in the plane in which are BD, DA, becauſe any three ſtraight lines which meet one another are in one plane [e]. therefore AB, BD, DC are in one plane. and each of the angles ABD, BDC is a right angle; therefore AB is parallel [f] to CD. Wherefore if two ſtraight lines, &c. Q. E. D.

c. 8. 1.
d. 5. 11.
e. 2. 11.
f. 28. 1.

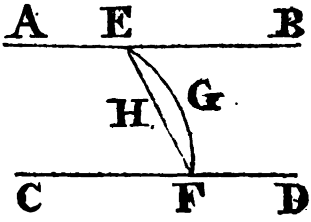
PROP. VII. THEOR.

See N.

IF two ſtraight lines be parallel, the ſtraight line drawn from any point in the one to any point in the other is in the ſame plane with the parallels.

Let AB, CD be parallel ſtraight lines, and take any point E in the one, and the point F in the other. the ſtraight line which joins E and F is in the ſame plane with the parallels.

If not, let it be, if poſſible, above the plane, as EGF; and in the plane ABCD in which the parallels are, draw the ſtraight line EHF from E to F; and ſince EGF alſo is a ſtraight line, the two ſtraight lines EHF, EGF include a ſpace betwixt them, which is impoſſible [a]. Therefore the ſtraight line joining the points E, F is not above the plane in which the parallels AB, CD are, and is therefore in that plane. Wherefore if two ſtraight lines, &c. Q. E. D.



a. 10. Ax. 1.

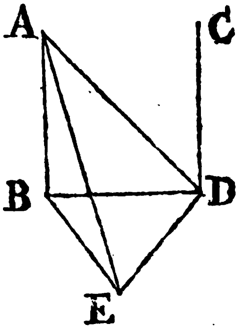
PROP. VIII. THEOR.

See N.

IF two ſtraight lines be parallel, and one of them is at right angles to a plane; the other alſo ſhall be at right angles to the ſame plane.

Let AB, CD be two parallel ſtraight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the ſame plane.

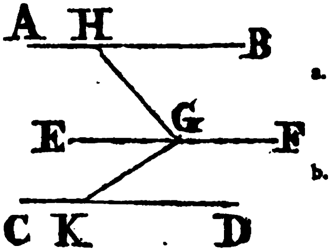
Let AB, CD, meet the plane in the points B, D, and join BD. therefore AB, CD, BD are in one plane. In the plane, to which AB is at right angles, draw DE at right angles to BD, and make DE equal to AB, and join BE, AE, AD. And becauſe AB is perpendicular to the plane, it is perpendicular to every ſtraight
a. 3. Def. 11. line which meets it, and is in that plane [a]. therefore each of the angles ABD, ABE, is a right angle. and becauſe the ſtraight line BD meets the parallel ſtraight lines AB, CD, the angles ABD,
b. 29. 1. CDB are together equal [b] to two right angles. and ABD is a right angle; therefore alſo CDB is a right angle, and CD perpendicular to BD. and becauſe AB is equal to DE, and BD common, the two AB, BD, are equal to the two ED, DB, and the angle ABD is equal to the angle EDB, becauſe each of them is a right
c. 4. 1. angle; therefore the baſe AD is equal [c] to the baſe BE. again, becauſe AB is equal to DE, and BE to AD; the two AB, BE are equal to the two ED, DA; and the baſe AE is common to the triangles ABE, EDA; wherefore the angle ABE is equal
d. 8. 1. [d] to the angle EDA. and ABE is a right angle; and therefore EDA is a right angle, and ED perpendicular to DA. but it is alſo perpendicular to BD; therefore ED
e. 4. 11. is perpendicular [e] to the plane which paſſes thro' BD, DA, and
f. 3. Def. 11. ſhall [f] make right angles with every ſtraight line meeting it in that plane. but DC is in the plane paſſing thro' BD, DA, becauſe all three are in the plane in which are the parallels AB, CD. wherefore ED is at right angles to DC; and therefore CD is at right angles to DE. but CD is alſo at right angles to DB; CD then is at right angles to the two ſtraight lines DE, DB in the point of their interſection D; and therefore is at right angles to the plane paſſing thro' DE, DB, which is the ſame plane to which AB is at right angles. Therefore if two ſtraight lines, &c. Q. E. D.



PROP. IX. THEOR.

TWO ſtraight lines which are each of them parallel to the ſame ſtraight line, and not in the ſame plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the ſame plane with it; AB ſhall be parallel to CD.



In EF take any point G, from which draw, in the plane paſſing thro' EF, AB, the ſtraight line GH at right angles to EF; and in the plane paſſing thro' EF, CD, draw GK at right angles to the ſame EF. and becauſe EF is perpendicular both to GH and GK, EF is perpendicular [a] to the plane HGK paſſing thro' them. a. 4. 11. and EF is parallel to AB; therefore AB is at right angles [b] to the plane HGK. b. 8. 11. for the ſame reaſon, CD is likewiſe at right angles to the plane HGK. therefore AB, CD are each of them at right angles to the plane HGK. but if two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel [c] to one another. therefore AB is parallel to CD. c. 6. 11. Wherefore two ſtraight lines, &c. Q. E. D.

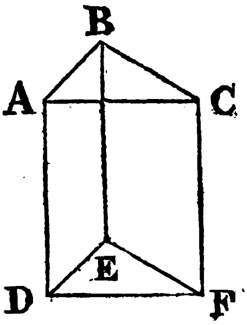
PROP. X. THEOR.

IF two ſtraight lines meeting one another be parallel to two others that meet one another, and are not in the ſame plane with the firſt two; the firſt two and the other two ſhall contain equal angles.

Let the two ſtraight lines AB, BC which meet one another be parallel to the two ſtraight lines DE, EF that meet one another, and are not in the ſame plane with AB, BC. The angle ABC is equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF. becauſe BA is equal and parallel to ED, there-

fore AD is [a] both equal and parallel to
BE. For the ſame reaſon, CF is equal
a. 33. 1. and parallel to BE. Therefore AD and
CF are each of them equal and parallel
to BE. But ſtraight lines that are parallel
to the ſame ſtraight line, and not in
b. 9. 11. the ſame plane with it, are parallel [b] to
one another. Therefore AD is parallel
c. 1. Ax. 1. to CF; and it is equal [c] to it, and AC,
DF join them towards the ſame parts;
and therefore [a] AC is equal and parallel
to DF. and becauſe AB, BC are equal to DE, EF, and the baſe
d. 8. 1. AC to the baſe DF; the angle ABC is equal [d] to the angle DEF.
Therefore if two ſtraight lines, &c. Q. E. D.

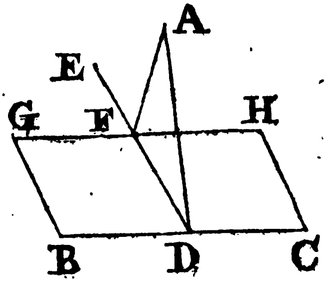


PROP. XI. PROB.

TO draw a ſtraight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a ſtraight line perpendicular to the plane BH.

In the plane draw any ſtraight line BC, and from the point A
a. 12. 1. draw [a] AD perpendicular to BC. If then AD be alſo perpendicular to the plane BH, the thing required is already done; but if it
b. 11. 1. be not, from the point D draw [b]
in the plane BH, the ſtraight line
DE at right angles to BC; and from
the point A draw [a] AF perpendicular
c. 31. 1. to DE; and thro' F draw [c] GH parallel
to BC. and becauſe BC is at
right angles to ED and DA, BC is
d. 4. 11. at right angles [d] to the plane paſ-
ſing thro' ED, DA. And GH is
parallel to BC; but if two ſtraight lines be parallel, one of which is
e. 8. 11. at right angles to a plane, the other ſhall be at right [e] angles to
the ſame plane; wherefore GH is at right angles to the plane thro'
f. 3. Def. 11. ED, DA, and is perpendicular [f] to every ſtraight line meeting it in
that plane. But AF, which is in the plane thro' ED, DA meets it.

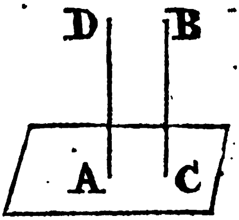


therefore GH is perpendicular to AF, and conſequently AF is perpendicular to GH. and AF is perpendicular to DE; therefore AF is perpendicular to each of the ſtraight lines GH, DE. but if a ſtraight line ſtands at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane paſſing through them. but the plane paſſing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH. therefore from the given point A above the plane BH, the ſtraight line AF is drawn perpendicular to that plane. Which was to be done.

PROP. XII. PROB.

TO erect a ſtraight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect a ſtraight line from the point A at right angles to the plane.



From any point B above the plane draw [a] BC perpendicular to it; and from A draw [b] AD parallel to BC. becauſe therefore AD, CB are two parallel ſtraight lines, and one of them BC is at right angles to the given plane, the other AD is alſo at right angles to it [c]. therefore a ſtraight line has been erected at right angles to a given plane from a point given in it. Which was to be done.

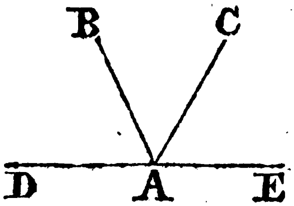
a. 11. 11.
b. 31. 1.
c. 8. 11.

PROP. XIII. THEOR.

FROM the ſame point in a given plane there cannot be two ſtraight lines at right angles to the plane, upon the ſame ſide of it. and there can be but one perpendicular to a plane from a point above the plane.

For, if it be poſſible, let the two ſtraight lines AB, AC be at right angles to a given plane from the ſame point A in the plane, and upon the ſame ſide of it; and let a plane paſs thro' BA, AC; the common ſection of this with the given plane is a ſtraight [a] line a. 3. 11.

paſſing through A. let DAE be their common ſection. therefore the ſtraight lines AB, AC, DAE are in one plane. and becauſe CA is at right angles to the given plane, it ſhall make right angles with every ſtraight line meeting it in that plane. but DAE which is in that plane meets CA; therefore CAE is a right angle. for the ſame reaſon BAE is a right angle. wherefore the angle CAE is equal to the angle BAE; and they are in one plane, which is impoſſible. Alſo, from a point above a plane there can be but one perpendicular to that plane; for if there could be two, they would be parallel [b] to one another, which is abſurd. Therefore from the ſame point, &c. Q. E. D. b. 6. 11.

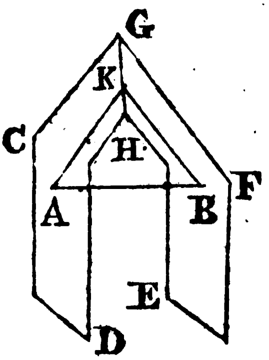


PROP. XIV. THEOR.

PLANES to which the ſame ſtraight line is perpendicular, are parallel to one another.

Let the ſtraight line AB be perpendicular to each of the planes CD, EF; theſe planes are parallel to one another.

If not, they ſhall meet one another when produced; let them meet; their common ſection ſhall be a ſtraight line GH, in which take any point K, and join AK, BK. then becauſe AB is perpendicular to the plane a. 3. Def. 11. EF, it is perpendicular [a] to the ſtraight line BK which is in that plane. therefore ABK is a right angle. for the ſame reaſon, BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK are equal to two right b. 17. 1. angles, which is impoſſible [b]. therefore the planes CD, EF though produced do c. 8. Def. 11. not meet one another; that is, they are parallel [c]. Therefore planes, &c. Q. E. D.

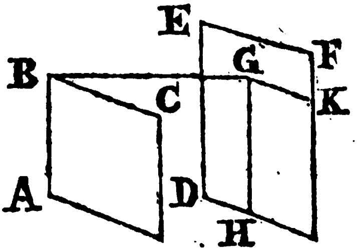


PROP. XV. THEOR.

IF two ſtraight lines meeting one another, be parallel to two ſtraight lines which meet one another, but are not in the ſame plane with the firſt two; the plane which paſſes through theſe is parallel to the plane paſſing through the others. See N.

Let AB, BC two ſtraight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the ſame plane with AB, BC. the planes through AB, BC, and DE, EF ſhall not meet though produced.

From the point B draw BG perpendicular [a] to the plane which paſſes through DE, EF, and let it meet that plane in G; and through G draw GH parallel [b] to ED, and GK parallel to EF. and becauſe BG is perpendicular to the plane through DE, EF, it ſhall make right angles with every ſtraight line meeting it in that plane [c]. but the ſtraight lines GH, GK in that plane meet it. therefore each of the angles BGH, BGK is a right angle. and becauſe BA is parallel [d] to GH (for each of them is parallel to DE, and they are not both in the ſame plane with it) the angles GBA, BGH are together equal [e] to two right angles. and BGH is a right angle, therefore alſo GBA is a right angle, and GB perpendicular to BA. for the ſame reaſon, GB is perpendicular to BC. ſince therefore the ſtraight line GB ſtands at right angles to the two ſtraight lines BA, BC, that cut one another in B; GB is perpendicular [f] to the plane through BA, BC. and it is perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC and DE, EF. but planes to which the ſame ſtraight line is perpendicular, are parallel [g] to one another. therefore the plane thro' AB, BC is parallel to the plane thro' DE, EF. Wherefore if two ſtraight lines, &c. Q. E. D.



a. 11. 11.
b. 31. 1.
c. 3. Def. 11.
d. 9. 11.
e. 29. 1.
f. 4. 11.
g. 14. 11.

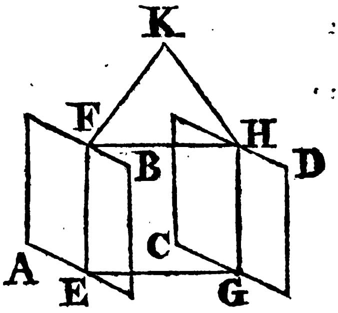
PROP. XVI. THEOR.

See N.

IF two parallel planes be cut by another plane, their common ſections with it are parallels.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common ſections with it be EF, GH. EF is parallel to GH.

For, if it is not, EF, GH ſhall meet, if produced, either on the ſide of FH, or EG. firſt, let them be produced on the ſide of FH, and meet in the point K, therefore ſince EFK is in the plane AB, every point in EFK is in that plane; and K is a point in EFK; therefore K is in the plane AB. for the ſame reaſon K is alſo in the plane CD. wherefore the planes AB, CD produced meet one another; but they do not meet, ſince they are parallel by the Hypotheſis. therefore the ſtraight lines EF, GH do not meet when produced on the ſide of FH. in the ſame manner it may be proved that EF, GH do not meet when produced on the ſide of EG. but ſtraight lines which are in the ſame plane and do not meet, though produced either way, are parallel. therefore EF is parallel to GH. Wherefore if two parallel planes, &c. Q. E. D.

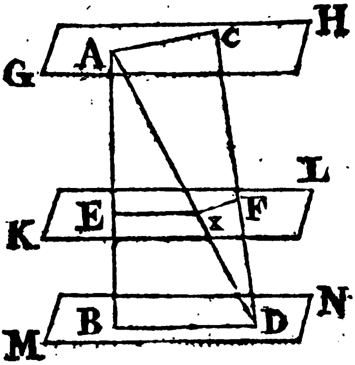


PROP. XVII. THEOR.

IF two ſtraight lines be cut by parallel planes, they ſhall be cut in the ſame ratio.

Let the ſtraight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D. as AE is to EB, ſo is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF. becauſe the two parallel planes KL, MN are cut by the plane EBDX, the common ſections EX, BD are parallel [a]. for the ſame reaſon, becauſe the two parallel planes GH, KL are cut by the plane AXFC, the common ſections AC, XF are parallel. a. 16. 11. and becauſe EX is parallel to BD, a ſide of the triangle ABD, as AE to EB, ſo is [b] AX to XD. b. 2. 6. again, becauſe XF is parallel to AC, a ſide of the triangle ADC, as AX to XD, ſo is CF to FD. and it was proved that AX is to XD, as AE to EB. therefore [c] as AE to EB, ſo is CF to FD. c. 11. 5. Wherefore if two ſtraight lines, &c. Q. E. D.

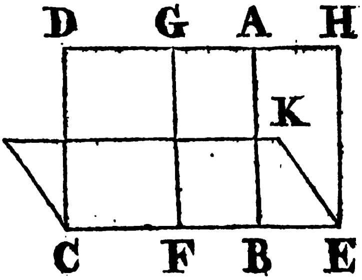


PROP. XVIII. THEOR.

IF a ſtraight line be at right angles to a plane, every plane which paſſes thro' it ſhall be at right angles to that plane.

Let the ſtraight line AB be at right angles to the plane CK. every plane which paſſes through AB ſhall be at right angles to the plane CK.

Let any plane DE paſs through AB, and let CE be the common ſection of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE. and becauſe AB is perpendicular to the plane CK, therefore it is alſo perpendicular to every ſtraight line in that plane meeting it [a]. a. 3. Def. 11. and conſequently it is perpendicular to CE, wherefore ABF is a right angle; but GFB is likewiſe a right angle; therefore AB is parallel [b] to FG. b. 28. 1. and AB is at right angles to the plane CK; therefore FG is alſo at right angles to the ſame plane [c]. c. 8. 11. but one plane is at right angles to another plane when the ſtraight lines drawn in one of the planes, at right angles



to their common ſection, are alſo at right angles to the other plane [d]; and any ſtraight line FG in the plane DE, which is at right angles to CE the common ſection of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which paſs through AB are at right angles to the plane CK. Therefore if a ſtraight line, &c. Q. E. D.

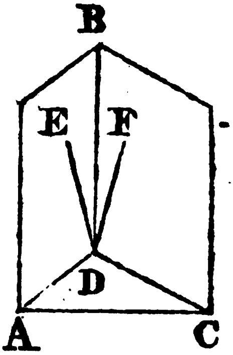
d. 4. Def. 11.

PROP. XIX. THEOR.

IF two planes cutting one another be each of them perpendicular to a third plane; their common ſection ſhall be perpendicular to the ſame plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common ſection of the firſt two, BD is perpendicular to the third plane.

If it be not, from the point D draw, in the plane AB, the ſtraight line DE at right angles to AD the common ſection of the plane AB with the third plane; and in the plane BC draw DF at right angles to CD the common ſection of the plane BC with the third plane. and becauſe the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common ſection, DE is perpendicular to the third plane [a]. in the ſame manner, it may be proved that DF is perpendicular to the third plane. wherefore from the point D two ſtraight lines ſtand at right angles to the third plane, upon the ſame ſide of it, which is impoſſible [b]. therefore from the point D there cannot be any ſtraight line at right angles to the third plane, except BD the common ſection of the planes AB, BC. BD therefore is perpendicular to the third plane. Wherefore if two planes, &c. Q. E. D.



a. 4. Def. 11.

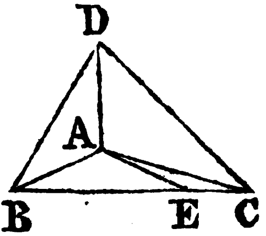
b. 13. 11.

PROP. XX. THEOR.

IF a ſolid angle be contained by three plane angles, any two of them are greater than the third. See N.

Let the ſolid angle at A be contained by the three plane angles BAC, CAD, DAB. any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. but if they are not, let BAC be that angle which is not leſs than either of the other two, and is greater than one of them DAB; and at the point A in the ſtraight line AB, make in the plane which paſſes through BA, AC, the angle BAE equal [a] to the angle DAB; and make AE a. 23. 1. equal to AD, and through E draw BEC cutting AB, AC in the points B, C, and join DB, DC. and becauſe DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, and the angle DAB is equal to the angle EAB. therefore the baſe DB is equal [b] b. 4. 1. to the baſe BE. and becauſe BD, DC are greater [c] than CB, and one of them BD has been proved equal c. 20. 1. to BE a part of CB, therefore the other DC is greater than the remaining part EC. and becauſe DA is equal to AE, and AC common, but the baſe DC greater than the baſe EC; therefore the angle DAC is greater [d] than the angle EAC; and, by the d. 25. 1. conſtruction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater than the angle BAC. but BAC is not leſs than either of the angles DAB, DAC, therefore BAC with either of them is greater than the other. Wherefore if a ſolid angle, &c. Q. E. D.

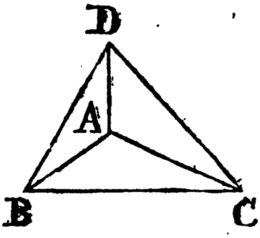


PROP. XXI. THEOR.

EVERY ſolid angle is contained by plane angles which together are leſs than four right angles.

Firſt, Let the ſolid angle at A be contained by three plane angles BAC, CAD, DAB. theſe three together are leſs than four right angles.

Take in each of the ſtraight lines AB, AC, AD any points B, C, D, and join BC, CD, DB. then, becauſe the ſolid angle at B is contained by the three plane angles CBA, ABD, DBC, any two of them are greater [a] than the third; therefore the angles CBA, ABD are greater than the angle DBC. for the ſame reaſon, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC. wherefore the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB. but the three angles DBC, BCD, CDB are equal to two right angles [b]. therefore the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles. and becauſe the three angles of each of the triangles ABC, ACD, ADB are equal to two right angles, therefore the nine angles of theſe three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA, DAC, ADB, DBA, BAD, are equal to ſix right angles. of theſe the ſix angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles. therefore the remaining three angles BAC, DAC, BAD which contain the ſolid angle at A, are leſs than four right angles.

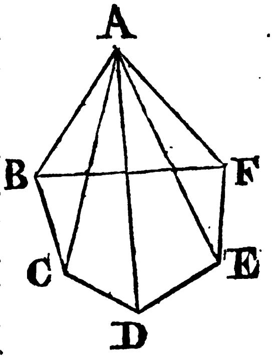


a. 20. 11.

b. 32. 1.

Next, Let the ſolid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; theſe together are leſs than four right angles.

Let the planes in which the angles are be cut by a plane, and let the common ſections of it with thoſe planes be BC, CD, DE, EF, FB. and becauſe the ſolid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater [a] than the third, the angles CBA, ABF are greater than the angle FBC. for the ſame reaſon, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the baſes of the triangles having the common vertex A, are greater than the third angle at the ſame point, which is one of the angles of the polygon BCDEF. therefore all the angles at the baſes of the triangles are together greater than all the



angles of the polygon. and becauſe all the angles of the triangles are together equal to twice as many right angles as there are triangles [b]; that is, as there are ſides in the polygon BCDEF; and that all the angles of the polygon together with four right angles are likewiſe equal to twice as many right angles as there are ſides in the polygon [c]; therefore all the angles of the triangles are equal to all the angles of the polygon together with four right angles. But all the angles at the baſes of the triangles are greater than all the angles of the polygon, as has been proved, wherefore the remaining angles of the triangles, viz. thoſe at the vertex, which contain the ſolid angle at A, are leſs than four right angles. Therefore every ſolid angle, &c. Q. E. D.

b. 32. 1.

c. 1. Cor. 32. 1.

PROP. XXII. THEOR.

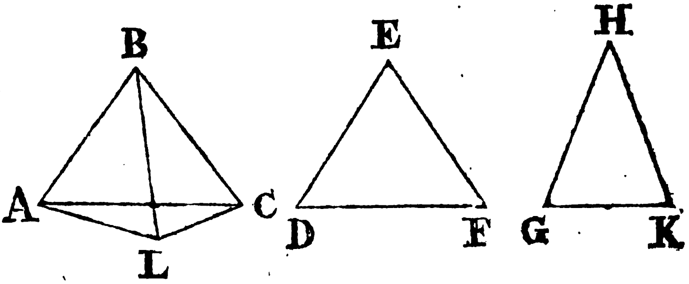
See N.

IF every two of three plane angles be greater than the third, and if the ſtraight lines which contain them be all equal; a triangle may be made of the ſtraight lines that join the extremities of thoſe equal ſtraight lines.

Let ABC, DEF, GHK be three plane angles, whereof every two are greater than the third, and are contained by the equal ſtraight lines AB, BC, DE, EF, GH, HK; if their extremities be joined by the ſtraight lines AC, DF, GK, a triangle may be made of three ſtraight lines equal to AC, DF, GK; that is, every two of them are together greater than the third.

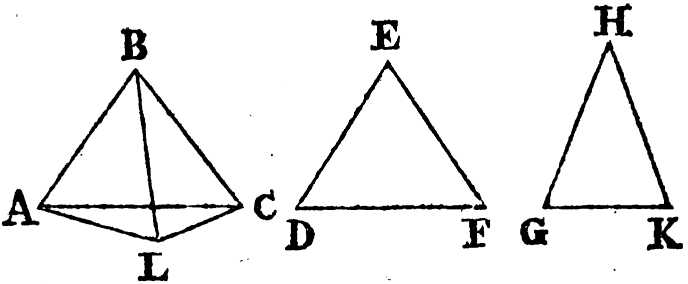
If the angles at B, E, H, are equal; AC, DF, GK are alſo equal[a], and any two of them greater than the third. but if the angles are

a. 4. 1.



not all equal, let the angle ABC be not leſs than either of the two at E, H; therefore the ſtraight line AC is not leſs than either of the

other two DF, GK [b]; and it is plain that AC together with either
of the other two muſt be greater than the third. alſo DF with GK
b. 4. or 24. 1. are greater than AC. for, at the point B in the ſtraight line AB
c. 23. 1. make [c] the angle ABL equal to the angle GHK, and make BL
equal to one of the ſtraight lines AB, BC, DE, EF, GH, HK, and
join AL, LC. then becauſe AB, BL are equal to GH, HK, and
the angle ABL to the angle GHK, the baſe AL is equal to the
baſe GK. and becauſe the angles at E, H are greater than the angle
ABC, of which GHK is equal to ABL, therefore the remaining
angle at E is greater than the angle LBC. and becauſe the two ſides
LB, BC are equal to the two DE, EF, and that the angle DEF is
d. 24. 1. greater than the angle LBC, the baſe DF is greater [d] than the baſe
LC. and it has been proved that GK is equal to AL; therefore
DF and GK are greater than AL and LC. but AL and LC are
e. 20. 1. greater [e] than AC; much more then are DF and GK greater than
AC. Wherefore every two of the ſtraight lines AC, DF, GK
f. 22. 1. are greater than the third, and therefore a triangle may be made [f]
the ſides of which ſhall be equal to AC, DF, GK. Q. E. D.

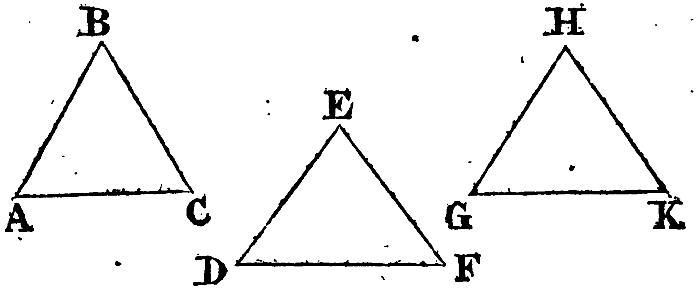


PROP. XXIII. PROB.

See N. TO make a ſolid angle which ſhall be contained by three given plane angles, any two of them being greater than the third, and all three together leſs than four right angles.

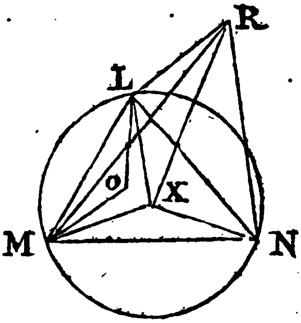
Let the three given plane angles be ABC, DEF, GHK, any two of which are greater than the third, and all of them together leſs than four right angles. It is required, to make a ſolid angle contained by three plane angles equal to ABC, DEF, GHK, each to each.

From the ſtraight lines containing the angles, cut off AB, BC, DE, EF, GH, HK all equal to one another; and join AC, DF, GK. then a triangle may be made [a] of three ſtraight lines equal a. 22. 11.



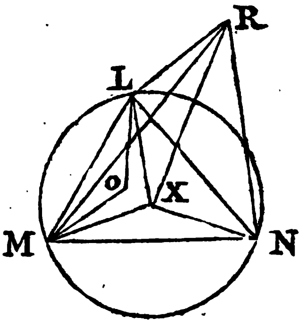
to AC, DF, GK. Let this be the triangle LMN [b], ſo that AC b. 22. 1. be equal to LM, DF to MN, and GK to LN; and about the triangle LMN deſcribe [c] a circle, and find its center X, which will c. 5. 4. either be within the triangle, or in one of its ſides, or without it.

Firſt, Let the center X be within the triangle, and join LX, MX, NX. AB is greater than LX. if not, AB muſt either be equal to, or leſs than LX; firſt, let it be equal. then becauſe AB is equal to LX, and that AB is alſo equal to BC, and LX to XM, AB and BC are equal to LX and XM, each to each; and the baſe AC is, by conſtruction, equal to the baſe LM; wherefore the angle ABC is equal to the angle LXM [d]. for the ſame reaſon, the d. 8. 1. angle DEF is equal to the angle MXN, and the angle GHK to the angle NXL. therefore the three angles ABC, DEF, GHK are equal to the three angles LXM, MXN, NXL. but the three angles LXM, MXN, NXL are equal to four right angles [e]; e. 2. Cor. 15. 1. therefore alſo the three angles ABC, DEF, GHK are equal to four right angles. but, by the hypotheſis, they are leſs than four right angles; which is abſurd. therefore AB is not equal to LX. but neither can AB be leſs than LX. for, if poſſible, let it be leſs, and upon the ſtraight line LM, on the ſide of it on which is the center X, deſcribe the triangle LOM, the ſides LO, OM of which are equal to AB, BC; and becauſe the baſe LM is equal to the baſe AC, the angle LOM



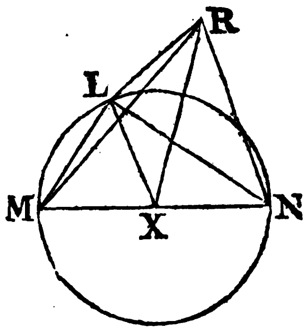
d. 8. 1.
f. 21. 1.

is equal to the angle ABC [d]. and AB, that is, LO, by the hypotheſis, is leſs than LX; wherefore LO, OM fall within the triangle LXM; for, if they fell upon its ſides, or without it, they would be equal to, or greater than LX, XM [f]. therefore the angle LOM, that is, the angle ABC is greater than the angle LXM [f]. in the ſame manner it may be proved, that the angle DEF is greater than the angle MXN, and the angle GHK greater than the angle NXL. therefore the three angles ABC, DEF, GHK are greater than three angles LXM, MXN, NXL; that is, than four right angles. but the ſame angles ABC, DEF, GHK are leſs than four right angles; which is abſurd. therefore AB is not leſs than LX, and it has been proved that it is not equal to LX; wherefore AB is greater than LX.



† 20. 1.

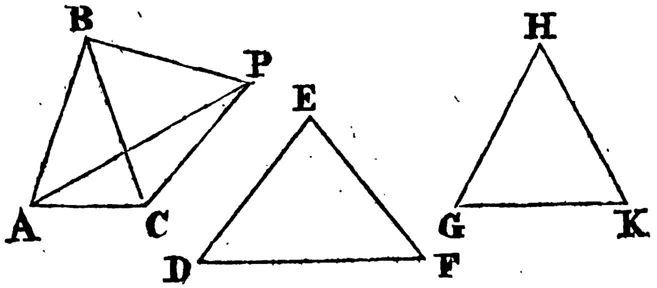
Next, Let the center X of the circle fall in one of the ſides of the triangle, viz. in MN, and join XL. in this caſe alſo AB is greater than LX. if not, AB is either equal to LX or leſs than it. firſt, let it be equal to LX. therefore AB and BC, that is, DE and EF, are equal to MX and XL, that is to MN. but, by the conſtruction, MN is equal to DF; therefore DE, EF are equal to DF, which is impoſſible †. wherefore AB is not equal to LX; nor is it leſs; for then, much more, an abſurdity would follow. therefore AB is greater than LX.



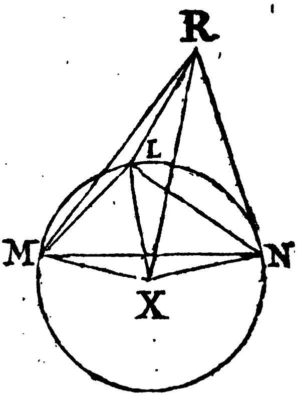
But let the center X of the circle fall without the triangle LMN, and join LX, MX, NX. In this caſe likewiſe AB is greater than LX. if not, it is either equal to, or leſs than LX. firſt, let it be equal; it may be proved, in the ſame manner, as in the firſt caſe, that the angle ABC is equal to the angle MXL, and GHK to LXN; therefore the whole angle MXN is equal to the two angles ABC. GHK. but ABC and GHK are together greater than the angle DEF; therefore alſo the angle MXN is greater than DEF.

and becauſe DE, EF are equal to MX, XN, and the baſe DF to the baſe MN, the angle MXN is equal [d] to the angle DEF. and it has been proved that it is greater than DEF, which is abſurd. therefore AB is not equal to LX. nor yet is it leſs; for then, as has been proved in the firſt caſe, the angle ABC is greater than the angle MXL, and the angle GHK greater than the angle LXN. at the point B in the ſtraight line CB make the angle CBP equal to the angle GHK, and make BP equal to HK, and join CP, AP.

d. 8. 1.



and becauſe CB is equal to GH; CB, BP are equal to GH, HK, each to each, and they contain equal angles; wherefore the baſe CP is equal to the baſe GK, that is, to LN. and in the Iſoſceles triangles ABC, MXL, becauſe the angle ABC is greater than the angle MXL, therefore the angle MLX at the baſe is greater [g] than the angle ACB at the baſe. for the ſame reaſon, becauſe the angle GHK, or CBP, is greater than the angle LXN, the angle XLN is greater than the angle BCP. therefore the whole angle MLN is greater than the whole angle ACP. and becauſe ML, LN are equal to AC, CP, each to each, but the angle MLN greater than the angle ACP, the baſe MN is greater [h] than the baſe AP. and MN is equal to DF; therefore alſo DF is greater than AP. Again, becauſe DE, EF are equal to AB, BP, but the baſe DF greater than the baſe AP, the angle DEF is greater [k] than the angle ABP. and ABP is equal to the two angles ABC, CBP, that is to the two angles ABC, GHK; therefore the angle DEF is greater than the two angles



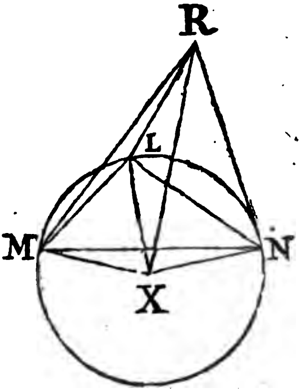
g. 32. 1.

h. 24. 1.

k. 25. 1.

ABC, GHK; but it is alſo leſs than theſe; which is impoſſible. therefore AB is not leſs than LX; and it has been proved that it is not equal to it; therefore AB is greater than LX.

l. 12. 11. From the point X erect [l] XR at right angles to the plane of the circle LMN. and becauſe it has been proved in all the caſes, that AB is greater than LX, find a ſquare equal to the exceſs of the ſquare of AB above the ſquare of LX, and make RX equal to its ſide, and join RL, RM, RN. becauſe RX is perpendicular to the plane of the m. 3. Def. 11. circle LMN, it is [m] perpendicular to each of the ſtraight lines LX, MX, NX. and becauſe LX is equal to MX, and XR common, and at right angles to each of them, the baſe RL is equal to the baſe RM. for the ſame reaſon, RN is equal to each of the two RL, RM. therefore the three ſtraight lines RL, RM, RN are all equal. and becauſe the ſquare of XR is equal to the exceſs of the ſquare of AB above the ſquare of LX; therefore the ſquare of AB is equal n. 47. 1. to the ſquares of LX, XR. but the ſquare of RL is equal [n] to the ſame ſquares, becauſe LXR is a right angle. therefore the ſquare of AB is equal to the ſquare of RL, and the ſtraight line AB to RL. but each of the ſtraight lines BC, DE, EF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL. wherefore AB, BC, DE, EF, GH, HK are each of them equal to each of the ſtraight lines RL, RM, RN. and becauſe RL, RM, are equal to AB, BC, and the baſe LM to the baſe AC; the angle LRM o. 8. 1. is equal [o] to the angle ABC. for the ſame reaſon, the angle MRN is equal to the angle DEF, and NRL to GHK. Therefore there is made a ſolid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK, each to each. Which was to be done.



PROP. A. THEOR.

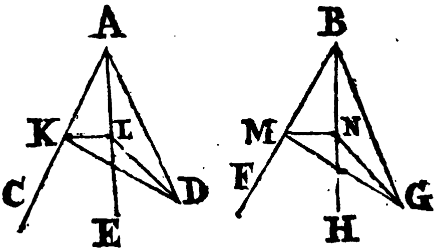
IF each of two ſolid angles be contained by three plane angles equal to one another, each to each; the planes in which the equal angles are have the ſame inclination to one another.

See N.

Let there be two ſolid angles at the points A, B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three plane angles FBG, FBH, HBG; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG. the planes in which the equal angles are, have the ſame inclination to one another.

In the ſtraight line AC take any point K, and in the plane CAD from K draw the ſtraight line KD at right angles to AC, and in the plane CAE the ſtraight line KL at right angles to the ſame AC. therefore the angle DKL is the inclination [a] of the plane CAD to the plane CAE. in BF take BM equal to AK, and from the point M draw, in the planes FBG, FBH, the ſtraight lines MG, MN at right angles to BF; therefore the angle GMN is the inclination [a] of the plane FBG to the plane FBH. join LD, NG; and becauſe in the triangles KAD, MBG, the angles KAD, MBG are equal, as alſo the right angles AKD, BMG, and that the ſides AK, BM, adjacent to the equal angles, are equal to one another, therefore KD is equal [b] to MG, and AD to BG. for the ſame reaſon, in the triangles KAL, MBN, KL is equal to MN, and AL to BN. and in the triangles LAD, NBG, LA, AD are equal to NB, BG, and they contain equal angles; therefore the baſe LD is equal [c] to the baſe NG. laſtly, in the triangles KLD, MNG, the ſides DK, KL are equal to GM, MN; and the baſe LD to the baſe NG; therefore the angle DKL is equal [d] to the angle GMN. but the angle DKL is the inclination of the plane CAD to the plane CAE,

a. 6. Def. 11.
b. 26. 1.
c. 4. 1.
d. 8. 1.



and the angle GMN is the inclination of the plane FBG to the plane FBH, which planes have therefore the ſame inclination [e] to one another. and in the ſame manner it may be demonſtrated, that the other planes in which the equal angles are, have the ſame inclination to one another. Therefore if two ſolid angles, &c. Q. E. D.

e. 7. Def. 11.

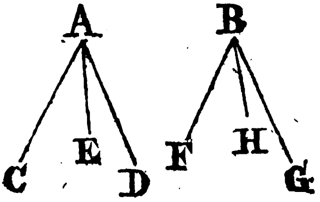
PROP. B. THEOR.

See N.

IF two ſolid angles be contained, each by three plane angles which are equal to one another, each to each, and alike ſituated; theſe ſolid angles are equal to one another.

Let there be two ſolid angles at A and B, of which the ſolid angle at A is contained by the three plane angles CAD, CAE, EAD; and that at B, by the three plane angles FBG, FBH, HBG; of which CAD is equal to FBG; CAE to FBH; and EAD to HBG. the ſolid angle at A, is equal to the ſolid angle at B.

Let the ſolid angle at A be applied to the ſolid angle at B; and firſt, the plane angle CAD being applied to the plane angle FBG, ſo as the point A may coincide with the point B, and the ſtraight line AC with BF; then AD coincides with BG, becauſe the angle CAD is equal to the angle FBG. and becauſe the inclination of the plane CAE to the plane CAD is equal [a] to the inclination of the plane FBH to the plane FBG, the plane CAE coincides with the plane FBH, becauſe the planes CAD, FBG coincide with one another. and becauſe the ſtraight lines AC, BF coincide, and that the angle CAE is equal to the angle FBH, therefore AE coincides with BH. and AD coincides with BG, wherefore the plane EAD coincides with the plane HBG. therefore the ſolid angle A coincides with the ſolid angle B, and conſequently they are equal [b] to one another. Q. E. D.



a. A. 11.
b. 8. Ax. 1.

PROP. C. THEOR.

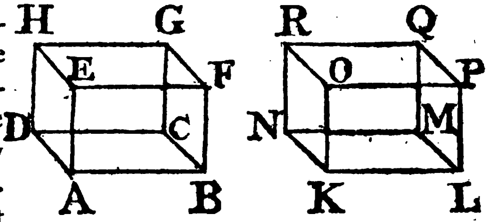
See N.

SOLID figures contained by the ſame number of equal and ſimilar planes alike ſituated, and having none of their ſolid angles contained by more than three plane angles; are equal and ſimilar to one another.

Let AG, KQ be two ſolid figures contained by the ſame number of ſimilar and equal planes, alike ſituated, viz. let the plane AC be ſimilar and equal to the plane KM; the plane AF to KP; BG to LQ; GD to QN; DE to NO; and laſtly, FH ſimilar and equal to PR. the ſolid figure AG is equal and ſimilar to the ſolid figure KQ.

a. B. 11.

Becauſe the ſolid angle at A is contained by the three plane angles BAD, BAE, EAD which, by the hypotheſis, are equal to the plane angles LKN, LKO, OKN which contain the ſolid angle at K, each to each; therefore the ſolid angle at A is equal [a] to the ſolid angle at K. in the ſame manner, the other ſolid angles of the figures are equal to one another. If then the ſolid figure AG be applied to the ſolid figure KQ, firſt, the plane figure AC being applied to the plane figure KM; the ſtraight line AB coinciding with KL, the figure AC muſt coincide with the figure KM, becauſe they are equal and ſimilar. therefore the ſtraight lines AD, DC, CB coincide with KN, NM, ML, each with each; and the points A, D, C, B with the points K, N, M, L. and the ſolid angle at A coincides [a] with the ſolid angle at K; wherefore the plane AF coincides with the plane KP, and the figure AF with the figure KP, becauſe they are equal and ſimilar to one another. therefore the ſtraight lines AE, EF, FB coincide with KO, OP, PL; and the points E, F, with the points O, P. In the ſame manner, the figure AH coincides with the figure KR, and the ſtraight line DH with NR, and the point H with the point R. and becauſe the ſolid angle at B is equal to the ſolid angle at L, it may be proved in the ſame manner, that the figure BG coincides



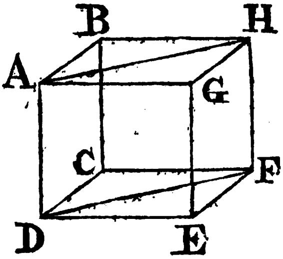
with the figure LQ, and the ſtraight line CG with MQ, and the point G with the point Q. ſince therefore all the planes and ſides of the ſolid figure AG coincide with the planes and ſides of the ſolid figure KQ, AG is equal and ſimilar to KQ. and in the ſame manner, any other ſolid figures whatever contained by the ſame number of equal and ſimilar planes, alike ſituated, and having none of their ſolid angles contained by more than three plane angles, may be proved to be equal and ſimilar to one another. Q. E. D.

PROP. XXIV. THEOR.

See N. IF a ſolid be contained by ſix planes, two and two of which are parallel; the oppoſite planes are ſimilar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE. its oppoſite planes are ſimilar and equal parallelograms.

Becauſe the two parallel planes, BG, CE are cut by the plane a. 16. 11. AC, their common ſections AB, CD are parallel [a]. again, becauſe the two parallel planes BF, AE are cut by the plane AC, their common ſections AD, BC are parallel [a]. and AB is parallel to CD; therefore AC is a parallelogram. in like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram. join AH, DF; and becauſe AB is parallel to DC, and BH to CF; the two ſtraight lines AB, BH, which meet one another, are parallel to DC and CF which meet one another and are not in the ſame plane b. 10. 11. with the other two; wherefore they contain equal angles [b]; the angle ABH is therefore equal to the angle DCF. and becauſe AB, BH are equal to DC, CF, and the angle ABH equal to the angle c. 4. 1. DCF, therefore the baſe AH is equal [c] to the baſe DF, and the triangle ABH to the triangle DCF. and the parallelogram BG is d. 34. 1. double [d] of the triangle ABH, and the parallelogram CE double of the triangle DCF; therefore the parallelogram BG is equal and ſimilar to the parallelogram CE. in the ſame manner, it may be proved that the parallelogram AC is equal and ſimilar to the pa-



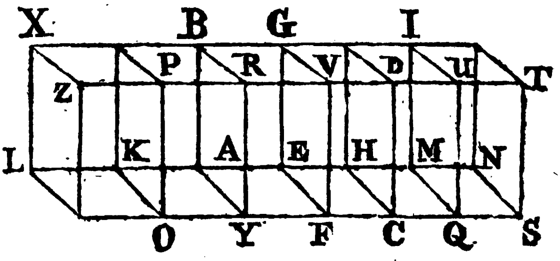
rallelogram GF, and the parallelogram AE to BF. Therefore if a ſolid, &c. Q. E. D.

PROP. XXV. THEOR.

IF a ſolid parallelepiped be cut by a plane parallel to two of its oppoſite planes; it divides the whole into two ſolids, the baſe of one of which ſhall be to the baſe of the other, as the one ſolid is to the other. See N.

Let the ſolid parallelepiped ABCD be cut by the plane EV which is parallel to the oppoſite planes AR, HD, and divides the whole into the two ſolids ABFV, EGCD; as the baſe AEFY of the firſt is to the baſe EHCF of the other, ſo is the ſolid ABFV to the ſolid EGCD.

Produce AH both ways, and take any number of ſtraight lines HM, MN each equal to EH, and any number AK, KL each equal to EA, and complete the parallelograms LO, KY, HQ, MS, and the ſolids LP, KR, HU, MT. then becauſe the ſtraight lines LK, KA, AE are all equal, the parallelograms LO, KY, AF are equal [a]. a. 36. 1.

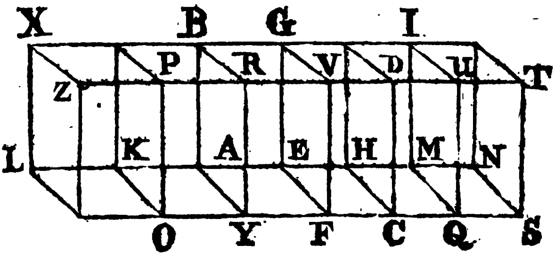


and likewiſe the parallelograms KX, KB, AG [a]; as alſo [b] the parallelograms LZ, KP, AR, becauſe they are oppoſite planes. for the ſame reaſon, the parallelograms EC, HQ, MS are equal [a]; and the parallelograms HG, HI, IN, as alſo [b] HD, MU, NT. therefore three planes of the ſolid LP, are equal and ſimilar to three planes of the ſolid KR, as alſo to three planes of the ſolid AV. but the three planes oppoſite to theſe three are equal and ſimilar [b] to them in the ſeveral ſolids, and none of their ſolid angles are contained by more than three plane angles. therefore the three ſolids LP, KR, AV are equal [c] to one another. for the ſame reaſon, the three ſolids ED, HU, MT are equal to one another. therefore what

b. 24. 11.

c. C. 11.

multiple ſoever the baſe LF is of the baſe AF, the ſame multiple is the ſolid LV of the ſolid AV. for the ſame reaſon, whatever multiple the baſe NF is of the baſe HF, the ſame multiple is the ſolid NV of the ſolid ED. and if the baſe LF be equal to the baſe NF,
c. C. 11. the ſolid LV is equal [c] to the ſolid NV; and if the baſe LF be greater than the baſe NF, the ſolid LV is greater than the ſolid NV; and if leſs, leſs. ſince then there are four magnitudes, viz.



the two baſes AF, FH, and the two ſolids AV, ED, and of the baſe AF and ſolid AV, the baſe LF and ſolid LV are any equimultiples whatever; and of the baſe FH and ſolid ED, the baſe FN and ſolid NV are any equimultiples whatever; and it has been proved, that if the baſe LF is greater than the baſe FN, the ſolid LV is greater than the ſolid NV; and if equal, equal; and if leſs,
d. 5. Def. 5. leſs. Therefore [d] as the baſe AF is to the baſe FH, ſo is the ſolid AV to the ſolid ED. Wherefore if a ſolid, &c. Q. E. D.

PROP. XXVI. PROB.

See N. AT a given point in a given ſtraight line, to make a ſolid angle equal to a given ſolid angle contained by three plane angles.

Let AB be a given ſtraight line, A a given point in it, and D a given ſolid angle contained by the three plane angles EDC, EDF, FDC. it is required to make at the point A in the ſtraight line AB a ſolid angle equal to the ſolid angle D.

a. 11. 11. In the ſtraight line DF take any point F, from which draw [a] FG perpendicular to the plane EDC, meeting that plane in G;
b. 23. 1. join DG, and at the point A in the ſtraight line AB make [b] the angle BAL equal to the angle EDC, and in the plane BAL make the angle BAK equal to the angle EDG; then make AK equal

to DG, and from the point K erect [c] KH at right angles to the plane BAL; and make KH equal to GF, and join AH. then the ſolid angle at A which is contained by the three plane angles BAL, BAH, HAL is equal to the ſolid angle at D contained by the three plane angles EDC, EDF, FDC.

c. 12. 11.

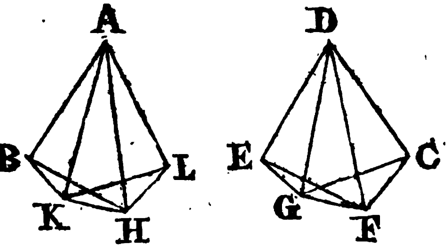
Take the equal ſtraight lines AB, DE, and join HB, KB, FE, GE. and becauſe FG is perpendicular to the plane EDC, it makes right angles [d] with every ſtraight line meeting it in that plane. therefore each of the angles FGD, FGE is a right angle. for the ſame reaſon, HKA, HKB are right angles. and becauſe KA, AB are equal to GD, DE, each to each, and contain equal angles, therefore the baſe BK is equal [e] to the baſe EG. and KH is equal to GF, and HKB, FGE are right angles, therefore HB is equal [e] to FE. again, becauſe AK, KH are equal to DG, GF, and contain right angles, the baſe AH is equal to the baſe DF; and AB is equal to DE; therefore HA, AB are equal to FD, DE, and the baſe HB is equal to the baſe FE; therefore the angle BAH is equal [f] to the angle EDF. for the ſame reaſon, the angle HAL is equal to the angle FDC. becauſe if AL and DC be made equal, and KL, HL, GC, FC be joined, ſince the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the conſtruction, equal; therefore the remaining angle KAL is equal to the remaining angle GDC. and becauſe KA, AL are equal to GD, DC, and contain equal angles, the baſe KL is equal [e] to the baſe GC. and KH is equal to GF, ſo that LK, KH are equal to CG, GF, and they contain right angles; therefore the baſe HL is equal to the baſe FC. again, becauſe HA, AL are equal to FD, DC, and the baſe HL to the baſe FC, the angle HAL is equal [f] to the angle FDC. therefore becauſe the three plane angles BAL, BAH, HAL which contain the ſolid angle at A, are equal to the three plane angles EDC, EDF, FDC which contain the ſolid angle at D, each to each, and are ſituated in the ſame order; the ſolid angle at A is equal [g] to the ſolid angle at D. Therefore at a given

d. 3. Def. 11.

e. 4. 1.

f. 8. 1.

g. B. 11.



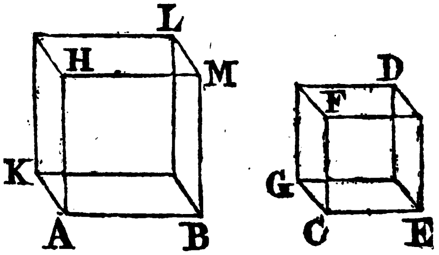
point in a given ſtraight line a ſolid angle has been made equal to a given ſolid angle contained by three plane angles. Which was to be done.

PROP. XXVII. PROB.

TO deſcribe from a given ſtraight line a ſolid parallelepiped ſimilar, and ſimilarly ſituated to one given.

Let AB be the given ſtraight line, and CD the given ſolid parallelepiped. It is required from AB to deſcribe a ſolid parallelepiped ſimilar, and ſimilarly ſituated to CD.

a. 26. 11. At the point A of the given ſtraight line AB make [a] a ſolid angle equal to the ſolid angle at C, and let BAK, KAH, HAB be the three plane angles which contain it, ſo that BAK be equal to the angle ECG, and KAH to GCF, and HAB to FCE. and as
b. 12. 6. EC to CG, ſo make [b] BA to AK, and as GC to CF, ſo make [b]
c. 22. 5. KA to AH; wherefore, ex aequali [c], as EC to CF, ſo is BA to AH. complete the parallelogram BH, and the ſolid AL. and becauſe, as EC to CG, ſo BA to AK, the ſides about the equal angles ECG, BAK are proportionals; therefore the parallelogram BK is ſimilar to EG. for the ſame reaſon the parallelogram KH is ſimilar to GF, and HB to FE. wherefore three parallelograms of the ſolid AL are ſimilar to three of the ſolid CD; and the three oppoſite
d. 24. 11. ones in each ſolid are equal [d] and ſimilar to theſe, each to each. alſo, becauſe the plane angles which contain the ſolid angles of the figures are equal, each to each, and ſituated in the ſame order,
e. B. 11. the ſolid angles are equal [e], each to each. Therefore the ſolid
f. 11. Def. 11 AL is ſimilar [f] to the ſolid CD. wherefore from a given ſtraight line AB a ſolid parallelepiped AL has been deſcribed ſimilar, and ſimilarly ſituated to the given one CD. Which was to be done.



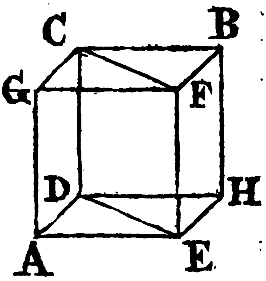
PROP. XXVIII. THEOR.

IF a ſolid parallelepiped be cut by a plane paſſing thro' the diagonals of two of the oppoſite planes; it ſhall be cut into two equal parts.

See N.

Let AB be a ſolid parallelepiped, and DE, CF the diagonals of the oppoſite parallelograms AH, GB, viz. thoſe which are drawn betwixt the equal angles in each. and becauſe CD, FE are each of them parallel to GA, and not in the ſame plane with it, CD, EF are parallel [a]; wherefore the diagonals CF, DE are in the plane in which the parallels are, and are themſelves parallels [b]. and the plane CDEF ſhall cut the ſolid AB into two equal parts.

a. 9. 11.
b. 16. 11.



Becauſe the triangle CGF is equal [c] to the triangle CBF, and the triangle DAE to DHE; and that the parallelogram CA is equal [d] and ſimilar to the oppoſite one BE; and the parallelogram GE to CH. therefore the priſm contained by the two triangles CGF, DAE, and the three parallelograms CA, GE, EC, is equal [e] to the priſm contained by the two triangles CBF, DHE, and the three parallelograms BE, CH, EC; becauſe they are contained by the ſame number of equal and ſimilar planes, alike ſituated, and none of their ſolid angles are contained by more than three plane angles. Therefore the ſolid AB is cut into two equal parts by the plane CDEF. Q. E. D.

c. 34. 1.
d. 24. 11.
e. C. 11.

‘N. B. The inſiſting ſtraight lines of a parallelepiped, mentioned in the next and ſome following Propoſitions, are the ſides of the parallelograms betwixt the baſe and the oppoſite plane parallel to it.’

PROP. XXIX. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of which are terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another.

See N.

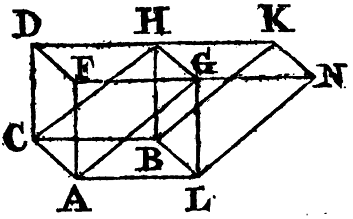
Book XI.

See the figures below.

Let the ſolid parallelepipeds AH, AK be upon the ſame baſe AB, and of the ſame altitude, and let their inſiſting ſtraight lines AF, AG, LM, LN; CD, CE, BH, BK be terminated in the ſame ſtraight lines FN, DK. the ſolid AH is equal to the ſolid AK.

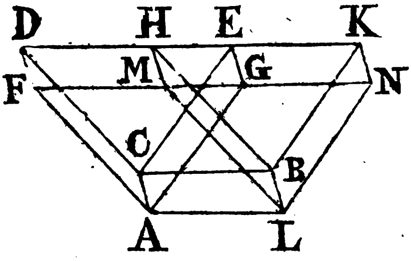
Firſt, Let the parallelograms DG, HN which are oppoſite to the baſe AB have a common ſide HG. then becauſe the ſolid AH is cut by the plane AGHC paſſing thro' the diagonals AG, CH of the oppoſite planes ALGF, CBHD, AH is cut into two equal parts [a] by the plane AGHC. therefore the ſolid AH is double of the priſm which is contained by the triangles ALG, CBH. for the ſame reaſon, becauſe the ſolid AK is cut by the plane LGHB thro' the diagonals LG, BH of the oppoſite planes ALNG, CBKH, the ſolid AK is double of the ſame priſm which is contained by the triangles ALG, CBH. Therefore the ſolid AH is equal to the ſolid AK.

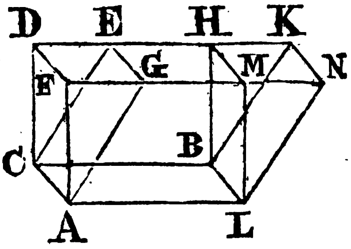
a. 28. 11.



But let the parallelograms DM, EN oppoſite to the baſe have no common ſide. then becauſe CH, CK are parallelograms, CB is equal [b] to each of the oppoſite ſides DH, EK; wherefore DH is equal to EK. add, or take away the common part HE; then DE is equal to HK. wherefore alſo the triangle CDE is equal [c] to the triangle BHK. and the parallelogram DG is equal [d] to the parallelogram HN. for the ſame reaſon, the triangle AFG is equal to the triangle LMN, and the parallelogram CF is equal [e] to the parallelogram BM, and CG to BN; for they are oppoſite. Therefore the priſm which is contained by the two triangles AFG, CDE, and the three parallelograms AD, DG, GC is equal [f] to the priſm contained by the two triangles LMN, BHK, and the three parallelograms BM, MK, KL. If therefore the priſm LMN, BHK be

b. 34. 1.
c. 38. 1.
d. 36. 1.
e. 24. 11.
f. C. 11.





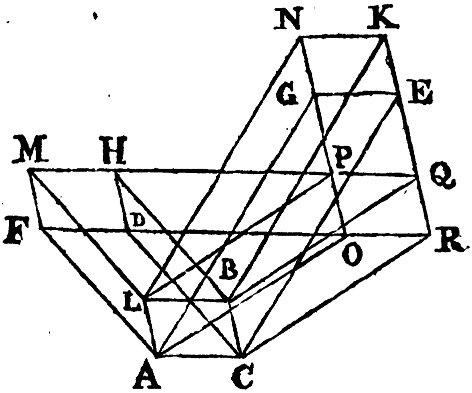
taken from the ſolid of which the baſe is the parallelogram AB, and in which FDKN is the one oppoſite to it; and if from this ſame ſolid there be taken the priſm AFG, CDE; the remaining ſolid, viz. the parallelepiped AH, is equal to the remaining parallelepiped AK. Therefore ſolid parallelepipeds, &c. Q. E. D.

PROP. XXX. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of which are not terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another. See N.

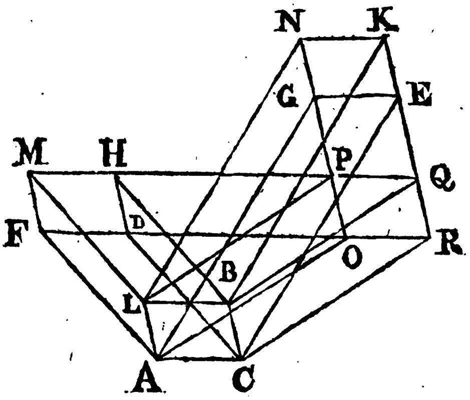
Let the parallelepipeds CM, CN be upon the ſame baſe AB, and of the ſame altitude, but their inſiſting ſtraight lines AF, AG, LM, LN, CD, CE, BH, BK not terminated in the ſame ſtraight lines. the ſolids CM, CN are equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR. and becauſe the plane LBHM is parallel to the oppoſite plane ACDF,



and that the plane LBHM is that in which are the parallels LB, MHPQ, in which alſo is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FDOR, in which alſo is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes. in like manner, becauſe the plane ALNG is parallel to the oppoſite plane CBKE, and that the plane ALNG is that in which are the parallels AL, OPGN, in which alſo is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, in which alſo is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes. and the planes ACBL, ORQP are parallel; therefore the ſolid CP is a parallelepiped. but the ſolid CM of which the baſe is ACBL, to which FDHM is the oppoſite parallelogram, is equal [a] to the ſolid CP of which the baſe is the parallelogram ACBL, to which ORQP is the one oppoſite; becauſe they are upon the ſame baſe, and their inſiſting ſtraight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the ſame ſtraight lines FR, MQ. and the ſolid CP is equal [a] to the ſolid CN, for they are upon the ſame baſe ACBL, and their inſiſting ſtraight lines AO, AG, LP, LN; CR, CE, BQ, BK are in the ſame ſtraight lines ON, RK. therefore the ſolid CM is equal to the ſolid CN. Wherefore ſolid parallelepipeds, &c. Q. E. D.

a. 29. 11.



PROP. XXXI. THEOR.

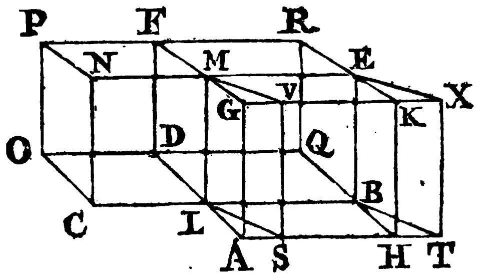
See N.

SOLID parallelepipeds which are upon equal baſes, and of the ſame altitude, are equal to one another.

Let the ſolid parallelepipeds AE, CF, be upon equal baſes AB, CD, and be of the ſame altitude; the ſolid AE is equal to the ſolid CF.

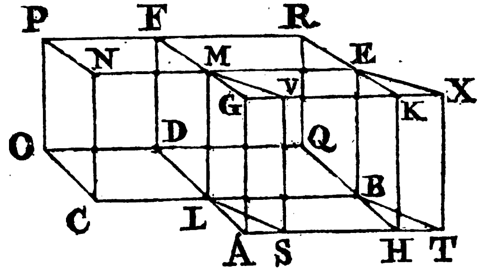
Firſt, let the inſiſting ſtraight lines be at right angles to the baſes AB, CD, and let the baſes be placed in the ſame plane, and ſo as

that the ſides CL, LB be in a ſtraight line; therefore the ſtraight line LM, which is at right angles to the plane in which the baſes are, in the point L, is common [a] to the two ſolids AE, CF; let the a. 13. 11.
other inſiſting lines of the ſolids be AG, HK, BE; DF, OP, CN. and firſt, let the angle ALB be equal to the angle CLD; then AL, LD are in a ſtraight line [b]. produce OD, HB, and let them b. 14. 1.
meet in Q, and complete the ſolid parallelepiped LR the baſe of which is the parallelogram LQ, and of which LM is one of its inſiſting ſtraight lines. therefore becauſe the parallelogram AB is equal to CD, as the baſe AB is to the baſe LQ, ſo is [c] the baſe c. 7. 5.
CD to the ſame LQ. and becauſe the ſolid parallelepiped AR is cut by the plane LMEB which is parallel to the oppoſite planes AK, DR; as the baſe AB is to the baſe LQ, ſo is [d] the ſolid d. 25. 11.
AE to the ſolid LR. for the ſame reaſon, becauſe the ſolid parallelepiped CR is cut by the plane LMFD which is parallel to the oppoſite planes CP, BR; as the baſe CD to the baſe LQ, ſo is the ſolid CF to the ſolid LR. but as the baſe AB to the baſe LQ, ſo the baſe CD to the baſe LQ, as before was proved. therefore as the ſolid AE to the ſolid LR, ſo is the ſolid CF to the ſolid LR; and therefore the ſolid AE is equal [e] to the ſolid CF. e. 9. 5.

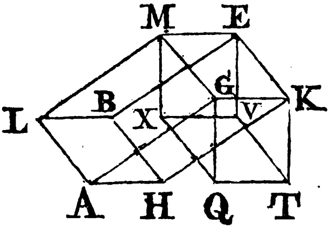


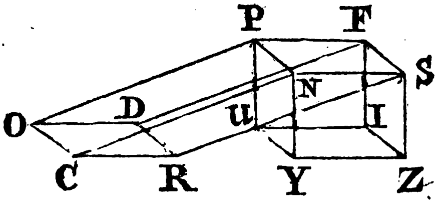
But let the ſolid parallelepipeds SE, CF be upon equal baſes SB, CD, and be of the ſame altitude, and let their inſiſting ſtraight lines be at right angles to the baſes; and place the baſes SB, CD in the ſame plane, ſo that CL, LB be in a ſtraight line; and let the angles SLB, CLD be unequal; the ſolid SE is alſo in this caſe equal to the ſolid CF. produce DL, TS until they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the ſolids AE, LR. therefore the ſolid AE, of which the baſe is the parallelogram LE, and AK the one oppoſite to it, is equal [f] to the ſolid SE, of which the baſe is LE, f. 29. 11.
and to which SX is oppoſite; for they are upon the ſame baſe LE, and of the ſame altitude, and their inſiſting ſtraight lines, viz. LA, LS, BH, BT; MG, MV, EK, EX are in the ſame ſtraight

lines AT, GX. and becauſe the parallelogram AB is equal[g] to SB, for they are upon the ſame baſe LB, and between the ſame paral- g. 35. 1. lels LB, AT; and that the baſe SB is equal to the baſe CD; therefore the baſe AB is equal to the baſe CD. and the angle ALB is equal to the angle CLD, therefore, by the firſt caſe, the ſolid AE is equal to the ſolid CF; but the ſolid AE is equal to the ſolid SE, as was demonſtrated; therefore the ſolid SE is equal to the ſolid CF.



But if the inſiſting ſtraight lines AG, HK, BE, LM; CN, RS, DF, OP, be not at right angles to the baſes AB, CD; in this caſe likewiſe the ſolid AE is equal to the ſolid CF. from the points G, K, E, M; N, S, F, P, draw the ſtraight lines GQ, KT, EV, MX; h. 11. 11. NY, SZ, FI, PU, perpendicular[h] to the plane in which are the baſes AB, CD; and let them meet it in the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX, XQ; YZ, ZI, IU, UY. then becauſe GQ, KT, are at right angles to the ſame plane, they are i. 6. 11. parallel[i] to one another. and MG, EK are parallels; therefore the planes MQ, ET of which one paſſes through MG, GQ, and the other through EK, KT which are parallel to MG, GQ, k. 15. 11. and not in the ſame plane with them, are parallel[k] to one another. for the ſame reaſon, the planes MV, GT are parallel to one another. therefore the ſolid QE is a parallelepiped. in like manner, it may be proved, that the ſolid YF is a parallelepiped. but, from what has been demonſtrated, the ſolid EQ is equal to the ſolid FY, becauſe they are upon equal baſes MK, PS, and of the ſame altitude, and





have their inſiſting ſtraight lines at right angles to the baſes. and the ſolid EQ is equal [l] to the ſolid AE; and the ſolid FY to the ſolid CF; becauſe they are upon the ſame baſes and of the ſame altitude. therefore the ſolid AE is equal to the ſolid CF. Wherefore ſolid parallelepipeds, &c. Q. E. D. l. 29. or 30. 11.

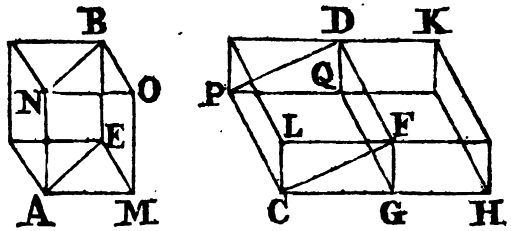
PROP. XXXII. THEOR.

SOLID parallelepipeds which have the ſame altitude, are to one another as their baſes. See N.

Let AB, CD be ſolid parallelepipeds of the ſame altitude. they are to one another as their baſes; that is, as the baſe AE to the baſe CF, ſo the ſolid AB to the ſolid CD.

To the ſtraight line FG apply the parallelogram FH equal [a] to AE, ſo that the angle FGH be equal to the angle LCG; and complete the ſolid parallelepiped GK upon the baſe FH, one of whoſe inſiſting lines is FD, whereby the ſolids CD, GK muſt be of the ſame altitude. therefore the ſolid AB is equal [b] to the ſolid GK, becauſe they are upon equal baſes AE, FH, and are of the ſame altitude. and becauſe the ſolid parallelepiped CK is cut by the plane DG which is parallel to its oppoſite planes, the baſe HF is [c] to the baſe FC, as the ſolid HD to the ſolid DC. but the baſe HF is equal to the baſe AE, and the ſolid GK to the ſolid AB. therefore as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD. Wherefore ſolid parallelepipeds, &c. Q. E. D.

a. Cor. 45. 1.
b. 31. 11.
c. 25. 11.



Cor. From this it is manifeſt that priſms upon triangular baſes, of the ſame altitude, are to one another as their baſes.

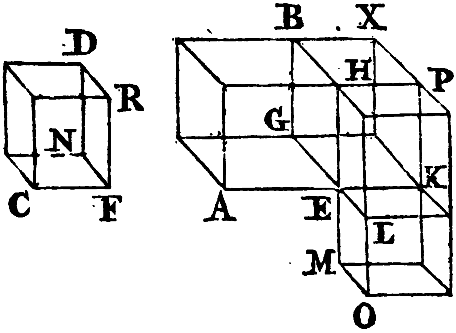
Let the priſms the baſes of which are the triangles AEM, CFG, and NBO, PDQ the triangles oppoſite to them, have the ſame altitude; and complete the parallelograms AE, CF, and the ſolid parallelepipeds AB, CD, in the firſt of which let MO, and in the other let GQ be one of the inſiſting lines. and becauſe the ſolid parallelepipeds AB, CD have the ſame altitude, they are to one

another as the baſe AE is to the baſe CF; wherefore the priſms, which are their halves [d], are to one another as the baſe AE to the
d. 28. 11. baſe CF; that is, as the triangle AEM to the triangle CFG.

PROP. XXXIII. THEOR.

SIMILAR ſolid parallelepipeds are one to another in the triplicate ratio of their homologous ſides.

Let AB, CD be ſimilar ſolid parallelepipeds, and the ſide AE homologous to the ſide CF. the ſolid AB has to the ſolid CD, the triplicate ratio of that which AE has to CF.



Produce AE, GE, HE, and in theſe produced take EK equal to CF, EL equal to FN, and EM equal to FR; and complete the parallelogram KL, and the ſolid KO. becauſe KE, EL are equal to CF, FN, and the angle KEL equal to the angle CFN, becauſe the angle AEG is equal to CFN, by reaſon that the ſolids AB, CD are ſimilar; therefore the parallelogram KL is ſimilar and equal to the parallelogram CN. for the ſame reaſon, the parallelogram MK is ſimilar and equal to CR, and alſo OE to FD. therefore three parallelograms of the ſolid KO are equal and ſimilar to three parallelograms of the ſolid CD. and the three oppoſite ones in each ſolid are
a. 24. 11. equal [a] and ſimilar to theſe. therefore the ſolid KO is equal
b. C. 11. [b] and ſimilar to the ſolid CD. complete the parallelogram GK, and complete the ſolids EX, LP upon the baſes GK, KL, ſo that EH be an inſiſting ſtraight line in each of them, whereby they muſt be of the ſame altitude with the ſolid AB. and becauſe the ſolids AB, CD are ſimilar, and by permutation, as AE is to CF, ſo is EG to FN, and ſo is EH to FR; and FC is equal to EK, and FN to EL, and FR to EM; therefore as AE to EK, ſo is EG to EL, and ſo is HE to EM. but as AE to
c. 1. 6. EK, ſo [c] is the parallelogram AG to the parallelogram GK; and

as GE to EL, ſo [c] is GK to KL; and as HE to EM, ſo [c] is PE to KM. therefore as the parallelogram AG to the parallelogram GK, ſo is GK to KL, and PE to KM. but as AG to GK, ſo [d] is the ſolid AB to the ſolid EX; and as GK to KL, ſo [d] is the ſolid EX to the ſolid PL; and as PE to KM, ſo [d] is the ſolid PL to the ſolid KO. and therefore as the ſolid AB to the ſolid EX, ſo is EX to PL, and PL to KO. but if four magnitudes be continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond. therefore the ſolid AB has to the ſolid KO, the triplicate ratio of that which AB has to EX. but as AB is to EX, ſo is the parallelogram AG to the parallelogram GK, and the ſtraight line AE to the ſtraight line EK. wherefore the ſolid AB has to the ſolid KO, the triplicate ratio of that which AE has to EK. and the ſolid KO is equal to the ſolid CD, and the ſtraight line EK is equal to the ſtraight line CF. Therefore the ſolid AB has to the ſolid CD, the triplicate ratio of that which the ſide AE has to the homologous ſide CF. Q. E. D.

c. 1. 6.
d. 25. 11.

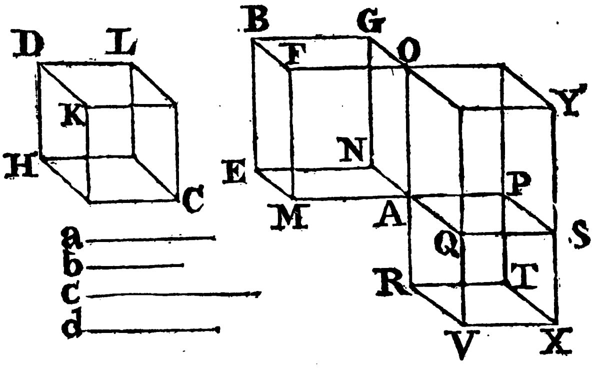
Cor. From this it is manifeſt, that if four ſtraight lines be continual proportionals, as the firſt is to the fourth, ſo is the ſolid parallelepiped deſcribed from the firſt to the ſimilar ſolid ſimilarly deſcribed from the ſecond; becauſe the firſt ſtraight line has to the fourth, the triplicate ratio of that which it has to the ſecond.

PROP. D. THEOR.

SOLID parallelepipeds contained by parallelograms equiangular to one another, each to each, that is, of which the ſolid angles are equal, each to each; have to one another the ratio which is the ſame with the ratio compounded of the ratios of their ſides. See N.

Let AB, CD be ſolid parallelepipeds, of which AB is contained by the parallelograms AE, AF, AG equiangular, each to each, to the parallelograms CH, CK, CL which contain the ſolid CD. the ratio which the ſolid AB has to the ſolid CD is the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH.

Produce MA, NA, OA to P, Q, R, ſo that AP be equal to DL, AQ to DK, and AR to DH; and complete the ſolid parallelepiped AX contained by the parallelograms AS, AT, AV ſimilar and equal to CH, CK, CL, each to each. therefore the a. C. 11. ſolid AX is equal [a] to the ſolid CD. complete likewiſe the ſolid AY the baſe of which is AS, and of which AO is one of its inſiſting ſtraight lines. Take any ſtraight line a, and as MA to AP, ſo make a to b; and as NA to AQ, ſo make b to c; and as OA to AR, ſo c to d. then becauſe the parallelogram AE is equiangular to AS, AE is to AS, as the ſtraight line a to c, as is demonſtrated in the 23. Prop. Book 6. and the ſolids AB, AY, being betwixt the parallel planes BOY, EAS, are of the ſame altitude. b. 32. 11. therefore the ſolid AB is to the ſolid AY, as [b] the baſe AE to the baſe AS; that is, as the ſtraight line a is to c. and the ſolid AY



c. 25. 11. is to the ſolid AX, as [c] the baſe OQ is to the baſe QR; that is, as the ſtraight line OA to AR; that is, as the ſtraight line c to the ſtraight line d. and becauſe the ſolid AB is to the ſolid AY, as a is to c, and the ſolid AY to the ſolid AX, as c is to d; ex aequali, the ſolid AB is to the ſolid AX, or CD which is equal to it, as the ſtraight line a is to d. but the ratio of a to d is ſaid to be compounded d. Def. A. 5. [d] of the ratios of a to b, b to c, and c to d, which are the ſame with the ratios of the ſides MA to AP, NA to AQ, and OA to AR, each to each. and the ſides AP, AQ, AR are equal to the ſides DL, DK, DH, each to each. Therefore the ſolid AB has to the ſolid CD the ratio which is the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH. Q. E. D.

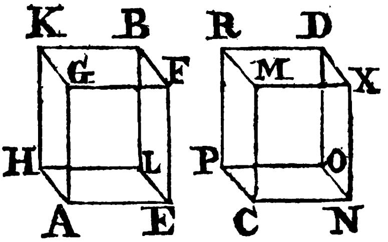
PROP. XXXIV. THEOR.

See N.

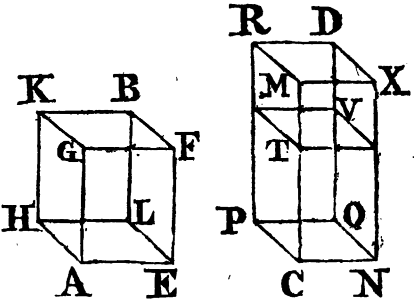
THE baſes and altitudes of equal ſolid parallelepipeds are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the ſolid parallelepipeds are equal.

Let AB, CD be equal ſolid parallelepipeds; their baſes are reciprocally proportional to their altitudes; that is, as the baſe EH is to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB.

Firſt, Let the inſiſting ſtraight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the baſes. as the baſe EH to the baſe NP, ſo is CM to AG. if the baſe EH be equal to the baſe NP, then becauſe the ſolid AB is likewiſe equal to the ſolid CD, CM ſhall be equal to AG. becauſe if the baſes EH, NP be equal, but the altitudes AG, CM be not equal, neither ſhall the ſolid AB be equal to the ſolid CD. but the ſolids are equal, by the hypotheſis. therefore the altitude CM is not unequal to the altitude AG; that is, they are equal. wherefore as the baſe EH to the baſe NP, ſo is CM to AG.

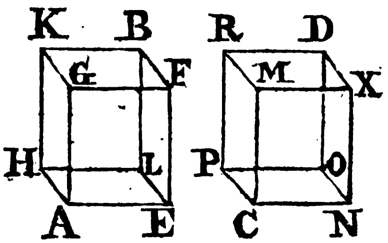


Next, Let the baſes EH, NP not be equal, but EH greater than the other. ſince then the ſolid AB is equal to the ſolid CD, CM is therefore greater than AG. for if it be not, neither alſo, in this caſe, would the ſolids AB, CD be equal, which, by the hypotheſis, are equal. Make then CT equal to AG, and complete the ſolid parallelepiped CV of which the baſe is NP, and altitude CT. Becauſe the ſolid AB is equal to the ſolid CD.

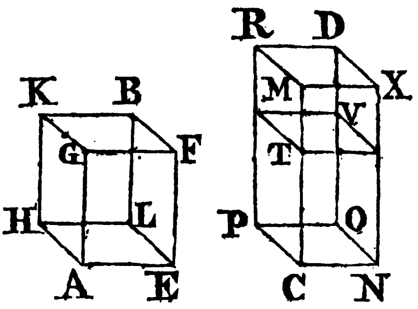


therefore the ſolid AB is to the ſolid CV, as [a] the ſolid CD to the ſolid CV. but as the ſolid AB to the ſolid CV, ſo [b] is the baſe EH
a. 7. 5. to the baſe NP; for the ſolids AB, CV are of the ſame altitude;
b. 32. 11. c. 25. 11. and as the ſolid CD to CV, ſo [c] is the baſe MP to the baſe PT,
d. 1. 6. and ſo [d] is the ſtraight line MC to CT; and CT is equal to AG. therefore as the baſe EH to the baſe NP, ſo is MC to AG. wherefore the baſes of the ſolid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

Let now the baſes of the ſolid parallelepipeds AB, CD be reciprocally proportional to their altitudes; viz. as the baſe EH to the baſe NP, ſo the altitude of the ſolid CD to the altitude of the ſolid AB; the ſolid AB is equal to the ſolid CD. let the inſiſting lines be, as before, at right angles to the baſes. then, if the baſe EH be equal to the baſe NP, ſince EH is to NP, as the altitude of the ſolid CD is to the altitude of the ſolid AB,
e. A. 5. therefore the altitude of CD is equal [e] to the altitude of AB. but ſolid parallelepipeds upon equal baſes, and of the ſame altitude are
f. 31. 11. equal [f] to one another; therefore the ſolid AB is equal to the ſolid CD.



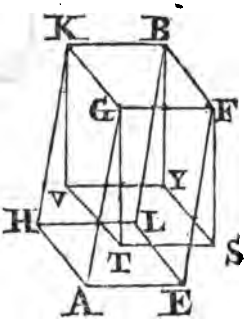
But let the baſes EH, NP be unequal, and let EH be the greater of the two. therefore, ſince as the baſe EH to the baſe NP, ſo is CM the altitude of the ſolid CD to AG the altitude of AB, CM is greater [e] than AG. again, take CT equal to AG, and complete, as before, the ſolid CV. and, becauſe the baſe EH is to the baſe NP, as CM to AG, and that AG is equal to CT, therefore the baſe EH is to the baſe NP, as MC to CT. but as the baſe EH is to NP, ſo [b] is the ſolid AB to the ſolid CV; for the ſolids AB, CV are of the ſame altitude; and as MC to CT, ſo is the baſe MP to the baſe PT, and the ſolid CD to the ſolid [c] CV, and

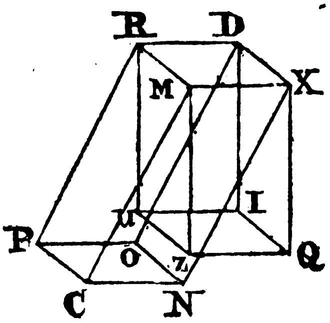


therefore as the ſolid AB to the ſolid CV, ſo is the ſolid CD to the ſolid CV; that is, each of the ſolids AB, CD has the ſame ratio to the ſolid CV; and therefore the ſolid AB is equal to the ſolid CD.

Second general Caſe. Let the inſiſting ſtraight lines FE, BL, GA, KH; XN, DO, MC, RP not be at right angles to the baſes of the ſolids; and from the points F, B, K, G; X, D, R, M draw perpendiculars to the planes in which are the baſes EH, NP, meeting thoſe planes in the points S, Y, V, T; Q, I, U, Z; and complete the ſolids FV, XU, which are parallelepipeds, as was proved in the laſt part of Prop. 31. of this Book. In this caſe likewiſe, if the ſolids AB, CD be equal, their baſes are reciprocally proportional to their altitudes, viz. the baſe EH to the baſe NP, as the altitude of the ſolid CD to the altitude of the ſolid AB. Becauſe the ſolid AB is equal to the ſolid CD, and that the ſolid BT is equal [g] to the ſolid BA, for they are upon the ſame baſe FK, and of the ſame altitude; and that the ſolid DC is equal [g] to the ſolid DZ, being upon the ſame baſe XR, and of the ſame altitude; therefore the ſolid BT is equal to the ſolid DZ. but the baſes are reciprocally proportional to the altitudes of equal ſolid parallelepipeds of which the inſiſting ſtraight lines are at right angles to their baſes, as before was proved. therefore as the baſe FK to the baſe XR, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT. and the baſe FK is equal to the baſe EH, and the baſe XR to the baſe NP. wherefore as the baſe EH to the baſe NP, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT. but the altitudes of the ſolids DZ, DC, as alſo of the ſolids BT, BA are the ſame. Therefore as the baſe EH to the baſe NP, ſo is the

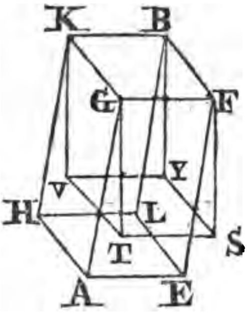
[g] 29. or 30. 11.

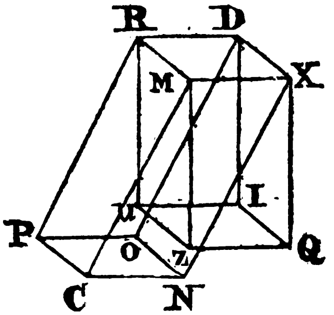




Book XI. altitude of the ſolid CD to the altitude of the ſolid AB; that is, the baſes of the ſolid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

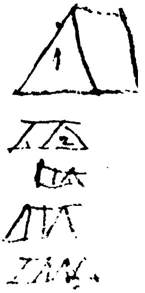
Next, Let the baſes of the ſolids AB, CD be reciprocally proportional to their altitudes, viz. the baſe EH to the baſe NP, as the altitude of the ſolid CD to the altitude of the ſolid AB; the ſolid AB is equal to the ſolid CD. the ſame conſtruction being made; becauſe as the baſe EH to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB; and that the baſe EH is equal to the baſe FK; and NP to XR; therefore the baſe FK is to the baſe XR, as the altitude of the ſolid CD to the altitude of AB. but the altitudes of the ſolids AB, BT are the ſame, as alſo of CD and DZ; therefore as the baſe FK to the baſe XR, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT. wherefore the baſes of the ſolids BT, DZ are reciprocally proportional to their altitudes; and their inſiſting ſtraight lines are at right angles to the baſes; wherefore, as was before proved, the g. 29. or 30. 11. ſolid BT is equal to the ſolid DZ. but BT is equal [g] to the ſolid BA, and DZ to the ſolid DC, becauſe they are upon the ſame baſes, and of the ſame altitude. Therefore the ſolid AB is equal to the ſolid CD. Q. E. D.



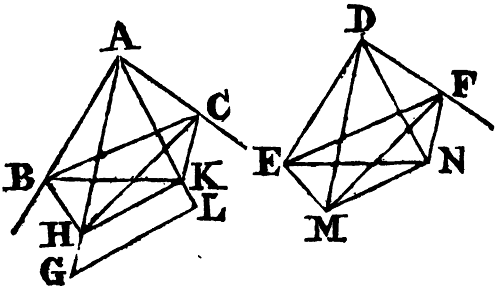


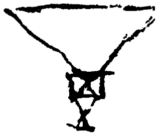
PROP. XXXV. THEOR.

IF from the vertices of two equal plane angles there be drawn two ſtraight lines elevated above the planes in which the angles are, and containing equal angles with the ſides of thoſe angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the firſt named angles are; and from the points in which they meet the planes, ſtraight lines be drawn to the vertices of the angles firſt named; theſe ſtraight lines ſhall contain equal angles with the ſtraight lines which are above the planes of the angles. See N.



Let BAC, EDF be two equal plane angles; and from the points A, D let the ſtraight lines AG, DM be elevated above the planes of the angles, making equal angles with their ſides, each to each; viz. the angle GAB equal to the angle MDE, and GAC to MDF;



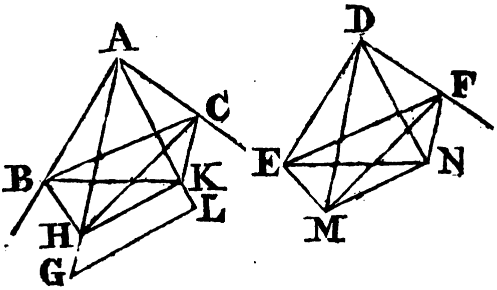


and in AG, DM let any points G, M be taken, and from them let perpendiculars GL, MN be drawn to the planes BAC, EDF meeting theſe planes in the points L, N; and join LA, ND. the angle GAL is equal to the angle MDN.

Make AH equal to DM, and thro' H draw HK parallel to GL. but GL is perpendicular to the plane BAC, wherefore HK is perpendicular [a] to the ſame plane. from the points K, N, to the ſtraight lines AB, AC, DE, DF, draw perpendiculars KB, KC, NE, NF; and join HB, BC, ME, EF. becauſe HK is perpendicular

a. 8. 11.

to the plane BAC, the plane HBK which paſſes through HK is at right angles [b] to the plane BAC; and AB is drawn in the plane BAC at right angles to the common ſection BK of the two planes; therefore AB is perpendicular [c] to the plane HBK, and makes right angles [d] with every ſtraight line meeting it in that plane. but BH meets it in that plane; therefore ABH is a right angle. for the ſame reaſon, DEM is a right angle, and is therefore equal to the angle ABH. and the angle HAB is equal to the angle MDE. therefore in the two triangles HAB, MDE there are two angles in one equal to two angles in the other, each to each, and one ſide equal to one ſide, oppoſite to one of the equal angles in each, viz. HA equal to DM; therefore the remaining ſides are equal [e], each to each. wherefore AB is equal to DE. In the ſame manner, if HC and MF be joined, it may be demonſtrated that AC is equal to DF. therefore ſince AB is equal to DE, BA and AC are equal to ED and DF; and the angle BAC is equal to the angle EDF; wherefore the baſe BC is equal [f] to the baſe EF, and the remaining angles to the remaining angles. the angle ABC is therefore equal to the angle DEF. and the right angle ABK is equal to the right angle DEN, whence the remaining angle CBK is equal to the remaining angle FEN. for the ſame reaſon, the angle BCK is equal to the angle EFN. therefore in the two triangles BCK, EFN there are two angles in one equal to two angles in the other, each to each, and one ſide equal to one ſide adjacent to the equal angles in each, viz. BC equal to EF; the other ſides therefore are equal to the other ſides; BK then is equal to EN. and AB is equal to DE; wherefore AB, BK are equal to DE, EN; and they contain right angles; wherefore the baſe AK is equal to the baſe DN. and ſince AH is equal to DM, the ſquare of AH is equal to the



b. 18. 11.
c. 4. Def. 11.
d. 3. Def. 11.
e. 26. 1.
f. 4. 1.

ſquare of DM. but the ſquares of AK, KH are equal to the ſquare [g] of AH, becauſe AKH is a right angle. and the ſquares of DN, NM are equal to the ſquare of DM, for DNM is a right angle. wherefore the ſquares of AK, KH are equal to the ſquares of DN, NM; and of thoſe the ſquare of AK is equal to the ſquare of DN. therefore the remaining ſquare of KH is equal to the remaining ſquare of NM; and the ſtraight line KH to the ſtraight line NM. and becauſe HA, AK are equal to MD, DN, each to each, and the baſe HK to the baſe MN, as has been proved; therefore the angle HAK is equal [h] to the angle MDN. Q. E. D.

g. 47. 1.

h. 8. 1.

Cor. From this it is manifeſt, that if from the vertices of two equal plane angles there be elevated two equal ſtraight lines containing equal angles with the ſides of the angles, each to each; the perpendiculars drawn from the extremities of the equal ſtraight lines to the planes of the firſt angles are equal to one another.

Another Demonſtration of the Corollary.

Let the plane angles BAC, EDF be equal to one another, and let AH, DM be two equal ſtraight lines above the planes of the angles, containing equal angles with BA, AC, ED, DF, each to each, viz. the angle HAB equal to MDE, and HAC equal to the angle MDF; and from H, M let HK, MN be perpendiculars to the planes BAC, EDF; HK is equal to MN.

Becauſe the ſolid angle at A is contained by the three plane angles BAC, BAH, HAC, which are, each to each, equal to the three plane angles EDF, EDM, MDF containing the ſolid angle at D; the ſolid angles at A and D are equal, and therefore coincide with one another; to wit, if the plane angle ABC be applied to the plane angle EDF, the ſtraight line AH coincides with DM, as was ſhewn in Prop. B. of this Book. and becauſe AH is equal to DM, the point H coincides with the point M. wherefore HK which is perpendicular to the plane BAC coincides with [i] MN which is perpendicular to the plane EDF, becauſe theſe planes coincide with one another. therefore HK is equal to MN. Q. E. D.

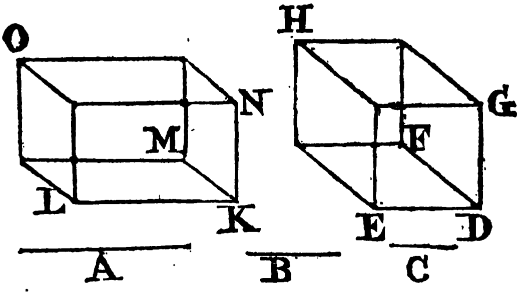
i. 13. 11.

PROP. XXXVI. THEOR.

See N. IF three ſtraight lines be proportionals, the ſolid parallelepiped deſcribed from all three as its ſides, is equal to the equilateral parallelepiped deſcribed from the mean proportional, one of the ſolid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the ſolid angles of the other figure.

Let A, B, C be three proportionals, viz. A to B, as B to C. The ſolid deſcribed from A, B, C is equal to the equilateral ſolid deſcribed from B, equiangular to the other.

Take a ſolid angle D contained by three plane angles EDF, FDG, GDE; and make each of the ſtraight lines ED, DF, DG equal to B, and complete the ſolid parallelepiped DH. make LK equal to A, and at the point K in the ſtraight line LK make [a] a ſolid angle contained by the three plane angles LKM, MKN, NKL equal to the angles EDF, FDG, GDE, each to each; and make KN equal to B, and KM equal to C; and complete the ſolid parallelepiped KO. and becauſe, as A is to B, ſo is B to C, and that A is equal to LK, and B to each of the ſtraight lines DE, DF, and C to KM; therefore LK is to ED, as DF to KM; that is, the ſides about the equal angles are reciprocally proportional; therefore the parallelogram LM is equal [b] to EF. and becauſe EDF, LKM are two equal plane angles, and the two equal ſtraight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their ſides; therefore the perpendiculars



[a] 26. 11.

[b] 14. 6.

from the points G, N, to the planes EDF, LKM are equal [c] to one another. therefore the ſolids KO, DH are of the ſame altitude; and they are upon equal baſes LM, EF, and therefore they are equal [d] to one another. but the ſolid KO is deſcribed from the three ſtraight lines A, B, C, and the ſolid DH from the ſtraight line B. If therefore three ſtraight lines, &c. Q. E. D.

c. Cor. 35. 11.
d. 31. 11.

PROP. XXXVII. THEOR.

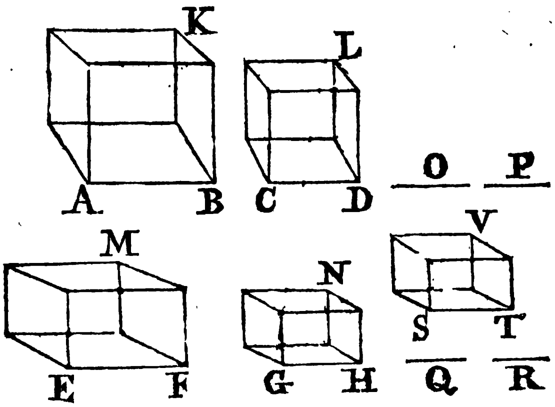
See N.

If four ſtraight lines be proportionals, the ſimilar ſolid parallelepipeds ſimilarly deſcribed from them ſhall alſo be proportionals. and if the ſimilar parallelepipeds ſimilarly deſcribed from four ſtraight lines be proportionals, the ſtraight lines ſhall be proportionals.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. as AB to CD, ſo EF to GH; and let the ſimilar parallelepipeds AK, CL, EM, GN be ſimilarly deſcribed from them. AK is to CL, as EM to GN.

Make [a] AB, CD, O, P continual proportionals, as alſo EF, GH, Q, R. and becauſe as AB is to CD, ſo EF to GH; CD is [b] to O, as GH to Q, and O to P, as Q to R; therefore, ex

a. 11. 6.
b. 11. 5.



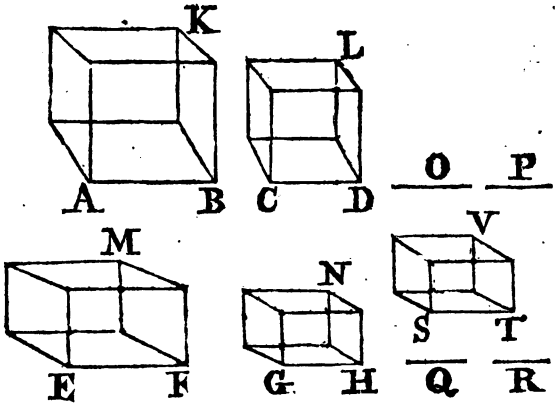
aequali [c], AB is to P, as EF to R. but as AB to P, ſo [d] is the ſolid AK to the ſolid CL; and as EF to R, ſo [d] is the ſolid EM to the ſolid GN. therefore [b] as the ſolid AK to the ſolid CL, ſo is the ſolid EM to the ſolid GN.

c. 22. 5.
d. Cor. 33. 11.

But let the ſolid AK be to the ſolid CL, as the ſolid EM to the ſolid GN. the ſtraight line AB is to CD, as EF to GH.

e. 27. 11. Take AB to CD, as EF to ST, and from ST deſcribe [e] a ſolid parallelepiped SV ſimilar and ſimilarly ſituated to either of the ſolids EM, GN. and becauſe AB is to CD, as EF to ST, and that from AB, CD the ſolid parallelepipeds AK, CL are ſimilarly deſcribed; and in like manner the ſolids EM, SV from the ſtraight lines EF, ST; therefore AK is to CL, as EM to SV. but, by the hypotheſis, AK is to CL, as EM to GN. therefore GN is equal [f] to SV. but it is likewiſe ſimilar and ſimilarly ſituated to SV; therefore the planes which contain the ſolids GN, SV are ſimilar and equal, and their homologous ſides GH, ST equal to one another. and becauſe as AB to CD, ſo EF to ST, and that ST is equal to GH; AB is to CD as EF to GH. Therefore, if four ſtraight lines, &c. Q. E. D.

f. 9. 5.



PROP. XXXVIII. THEOR.

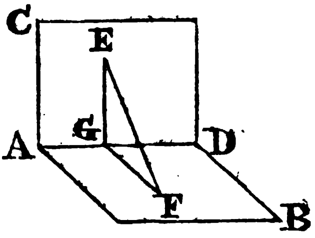
See N.

" IF a plane be perpendicular to another plane, and a " ſtraight line be drawn from a point in one of the " planes perpendicular to the other plane, this ſtraight " line ſhall fall on the common ſection of the planes.

" Let the plane CD be perpendicular to the plane AB, and let " AD be their common ſection; if any point E be taken in the " plane CD, the perpendicular drawn from E to the plane AB " ſhall fall on AD.

Book XI.

“For if it does not, let it, if poſſible, fall elſewhere, as EF; and let it meet the plane AB in the point F; and from F draw[a], in the plane AB, a perpendicular FG to DA, which is alſo perpendicular[b] to the plane CD; and join EG. then becauſe FG is perpendicular to the plane CD, and the ſtraight line EG, which is in that plane, meets it; therefore FGE is a right angle[c]. but EF is alſo at right angles to the plane AB; and therefore EFG is a right angle. wherefore two of the angles of the triangle EFG are equal together to two right angles; which is abſurd. therefore the perpendicular from the point E to the plane AB does not fall elſewhere than upon the ſtraight line AD. it therefore falls upon it. If therefore a plane, &c. Q. E. D.”



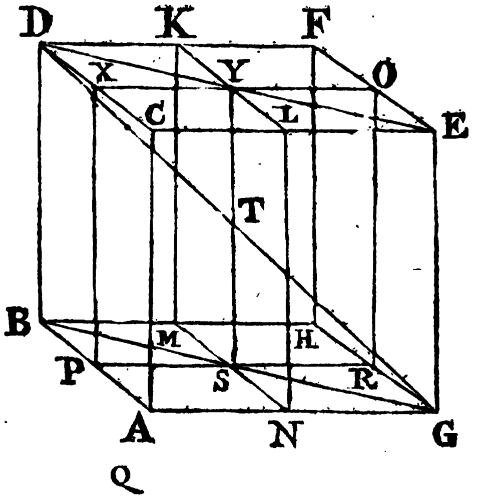
a. 12. 1.
b. 4. Def. 11.
c. 3. Def. 11.

PROP. XXXIX. THEOR.

See N.

IN a ſolid parallelepiped, if the ſides of two of the oppoſite planes be divided each into two equal parts, the common ſection of the planes paſſing through the points of diviſion, and the diameter of the ſolid parallelepiped cut each other into two equal parts.

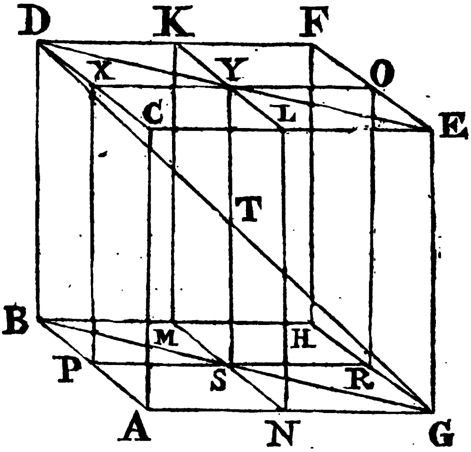
Let the ſides of the oppoſite planes CF, AH of the ſolid parallelepiped A F, be divided each into two equal parts in the points K, L, M, N; X, O, P, R; and join KL, MN, XO, P R. and becauſe DK, CL are equal and parallel, KL is parallel[a] to DC. for the ſame reaſon, MN is parallel to



a. 33. 1.

BA. and BA is parallel to DC; therefore becauſe KL, BA are each of them parallel to DC, and not in the ſame plane with it, KL
b. 9. 11. is parallel [b] to BA. and becauſe KL, MN are each of them parallel to BA, and not in the ſame plane with it, KL is parallel [b] to MN; wherefore KL, MN are in one plane. In like manner, it may be proved that XO, PR are in one plane. Let YS be the common ſection of the planes KN, XR; and DG the diameter of the ſolid parallelepiped AF. YS and DG do meet, and cut one another into two equal parts.

Join DY, YE, BS, SG. becauſe DX is parallel to OE, the al-
c. 29. 1. ternate angles DXY, YOE are equal [c] to one another. and becauſe DX is equal to OE, and XY to YO, and contain equal angles, the baſe DY
d. 4. 1. is equal [d] to the baſe YE, and the other angles are equal; therefore the angle XYD is equal to the angle OYE, and DYE is
e. 14. 1. a ſtraight [e] line. for the ſame reaſon BSG is a ſtraight line, and BS equal to SG. and becauſe CA is equal and parallel to DB, and alſo equal and parallel to EG; therefore DB is equal and parallel [b] to EG. and DE, BG join their extremities; therefore DE is equal and
a. 33. 1. parallel [a] to BG. and DG, YS are drawn from points in the one to points in the other, and are therefore in one plane. whence it is manifeſt that DG, YS muſt meet one another; let them meet in T. and becauſe DE is parallel to BG, the alternate angles EDT,
f. 15. 1. BGT are equal [c]; and the angle DTY is equal [f] to the angle GTS. therefore in the triangles DTY, GTS there are two angles in the one equal to two angles in the other, and one ſide equal to one ſide, oppoſite to two of the equal angles, viz. DY to GS; for they are the halves of DE, BG. therefore the remaining ſides
g. 26. 1. are equal [g], each to each. wherefore DT is equal to TG, and YT equal to TS. Therefore if in a ſolid, &c. Q. E. D.



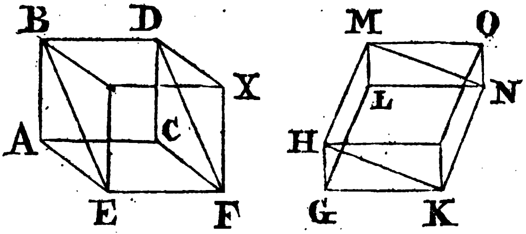
PROP. XL. THEOR.

IF there be two triangular priſms of the ſame altitude, the baſe of one of which is a parallelogram, and the baſe of the other a triangle; if the parallelogram be double of the triangle, the priſms ſhall be equal to one another.

Let the priſms ABCDEF, GHKLMN be of the ſame altitude, the firſt whereof is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN and the three parallelograms LH, HN, NG; and let one of them have a parallelogram AF, and the other a triangle GHK for its baſe; if the parallelogram AF be double of the triangle GHK, the priſm ABCDEF is equal to the priſm GHKLMN.

Complete the ſolids AX, GO; and becauſe the parallelogram AF is double of the triangle GHK; and the parallelogram HK double [a] of the ſame triangle; therefore the parallelogram AF is equal to HK. but ſolid parallelepipeds upon equal baſes, and of the ſame altitude are equal [b] to one another. therefore the ſolid AX is equal to the ſolid GO. and the priſm ABCDEF is half [c] of the ſolid AX; and the priſm GHKLMN half [c] of the ſolid GO. therefore the priſm ABCDEF is equal to the priſm GHKLMN. Wherefore if there be two, &c. Q. E. D.

a. 34. 1.
b. 31. 11.
c. 28. 11.



THE ELEMENTS OF EUCLID.

See N.

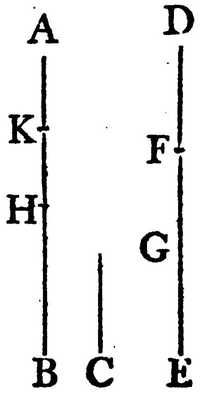
BOOK XII.

LEMMA I.

Which is the firſt Propoſition of the tenth Book, and is neceſſary to ſome of the Propoſitions of this Book.

IF from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half; and ſo on. there ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. if from AB there be taken more than its half, and from the remainder more than its half, and ſo on; there ſhall at length remain a magnitude leſs than C.



For C may be multiplied ſo as at length to become greater than AB. let it be ſo multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. from AB take BH greater than its half, and from the remainder AH take HK greater than its half, and ſo on until there be as many diviſions in AB as there are in DE. and let the diviſions AK, KH, HB be as many as the diviſions DF, FG, GE. and becauſe DE is greater than AB, and that EG taken from DE is leſs than its half, but BH taken from AB is greater than its half, therefore the remainder GD is greater than the remainder HA. again, becauſe GD is greater than HA, and that GF is the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK. and FD is equal to C, therefore C is greater than AK; that is, AK is leſs than C. Q. E. D.

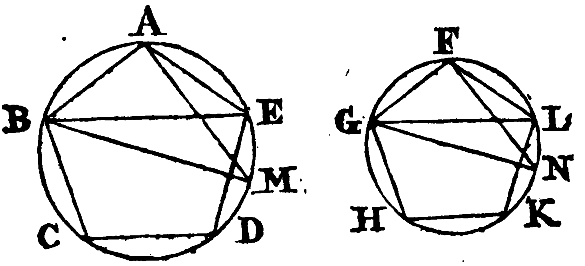
And if only the halves be taken away, the ſame thing may in the ſame way be demonſtrated.

PROP. I. THEOR.

SIMILAR polygons inſcribed in circles, are to one another as the ſquares of their diameters.

Let ABCDE, FGHKL be two circles, and in them the ſimilar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles. as the ſquare of BM is to the ſquare of GN, ſo is the polygon ABCDE to the polygon FGHKL.

Join BE, AM, GL, FN. and becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE is equal [a] to the angle a. 1. Def. 6.
GFL, and as BA to AE, ſo [a] is GF to FL. wherefore the two triangles BAE, GFL having one angle in one equal to one angle in the

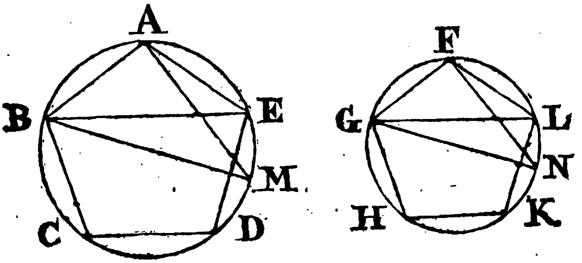


other, and the ſides about the equal angles proportionals, are equiangular [b]; and therefore the angle AEB is equal to the angle FLG. b. 6. 6.
but AEB is equal [c] to the angle AMB, becauſe they ſtand upon the c. 21. 3.
ſame circumference; and the angle FLG is, for the ſame reaſon, equal to the angle FNG. therefore alſo the angle AMB is equal to FNG. and the right angle BAM is equal to the right [d] angle GFN; d. 31. 3.
wherefore the remaining angles in the triangles ABM, FGN are equal, and they are equiangular to one another. therefore as BM to

Book XII.

GN, ſo [e] is BA to GF, and therefore the duplicate ratio of BM to GN, is the ſame [f] with the duplicate ratio of BA to GF. but the ratio of the ſquare of BM to the ſquare of GN, is the duplicate [g] ratio of that which BM has to GN; and the ratio of the polygon

e. 4. 6.
f. 10. Def. 5. and 22. 5.
g. 20. 6.



ABCDE to the polygon FGHKL is the duplicate [g] of that which BA has to GF. therefore as the ſquare of BM to the ſquare of GN, ſo is the polygon ABCDE to the polygon FGHKL. Wherefore ſimilar polygons, &c. Q. E. D.

PROP. II. THEOR.

See N.

CIRCLES are to one another as the ſquares of their diameters.

Let ABCD, EFGH be two circles, and BD, FH their diameters. as the ſquare of BD to the ſquare of FH, ſo is the circle ABCD to the circle EFGH.

For, if it be not ſo, the ſquare of BD ſhall be to the ſquare of FH, as the circle ABCD is to ſome ſpace either leſs than the circle EFGH, or greater than it *. Firſt, let it be to a ſpace S leſs than the circle EFGH; and in the circle EFGH deſcribe the ſquare EFGH. this ſquare is greater than half of the circle EFGH; becauſe if through the points E, F, G, H, there be drawn tangents to the circle, the ſquare EFGH is half [a] of the ſquare deſcribed

a. 41. 1.

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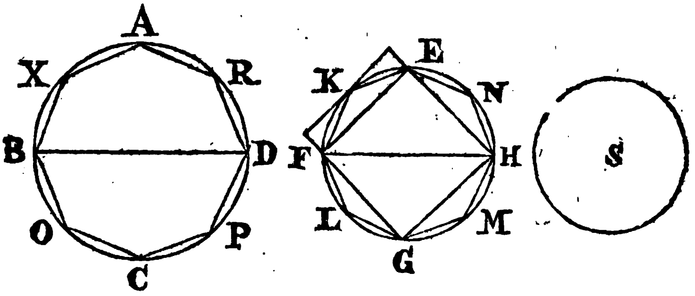
* For there is ſome ſquare equal to the circle ABCD; let P be the ſide of it. and to three ſtraight lines BD, FH and P, there can be a fourth proportional, let this be Q. therefore the ſquares of theſe four ſtraight lines are proportionals; that is, to the ſquares of BD, FH and the circle ABCD it is poſſible there may be a fourth proportional. Let this be S. and in like manner are to be underſtood ſome things in ſome of the following Propoſitions.

about the circle; and the circle is leſs than the ſquare deſcribed about it; therefore the ſquare EFGH is greater than half of the circle. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE. therefore each of the triangles EKF, FLG, GMH, HNE is greater than half of the ſegment of the circle it ſtands in; becauſe if ſtraight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the ſtraight lines EF, FG, GH, HE be completed; each of the triangles EKF, FLG, GMH, HNE ſhall be the half [a] of the parallelogram in which it is. but every ſegment is leſs than the parallelogram in which it is. wherefore each of the triangles EKF, FLG, GMH, HNE is greater than half the ſegment of the circle which contains it. and if theſe circumferences before named be divided each into two equal parts, and their extremities be joined by ſtraight lines, by continuing to do

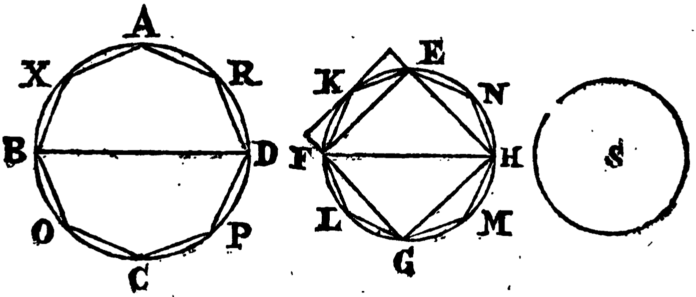
a. 41. 1.

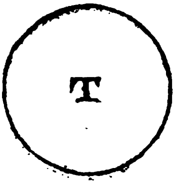


this, there will at length remain ſegments of the circle which together ſhall be leſs than the exceſs of the circle EFGH above the ſpace S. becauſe, by the preceding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and ſo on, there ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes. Let then the ſegments EK, KF, FL, LG, GM, MH, HN, NE be thoſe that remain and are together leſs than the exceſs of the circle EFGH above S. therefore the reſt of the circle, viz. the polygon EKFLGMHN is greater than the ſpace S. Deſcribe likewiſe in the circle ABCD the polygon AXBOCPDR ſimilar to the polygon EKFLGMHN. as, therefore, the ſquare of BD is to the ſquare of FH, ſo [b] is the polygon AXBOCPDR to the polygon EKFLGMHN. but the ſquare of BD is alſo to the ſquare of FH, as the circle ABCD

b. 1. 12.

is to the ſpace S. therefore as the circle ABCD is to the ſpace S, ſo is [c] the polygon AXBOCPDR to the polygon EKFLGMHN. but c. 11. 5. the circle ABCD is greater than the polygon contained in it; where- d. 14. 5. fore the ſpace S is greater [d] than the polygon EKFLGMHN. but it is likewiſe leſs, as has been demonſtrated; which is impoſſible. Therefore the ſquare of BD is not to the ſquare of FH, as the circle ABCD is to any ſpace leſs than the circle EFGH. in the ſame manner it may be demonſtrated that neither is the ſquare of FH to the ſquare of BD, as the circle EFGH is to any ſpace leſs than the circle ABCD. Nor is the ſquare of BD to the ſquare of FH, as the circle ABCD is to any ſpace greater than the circle EFGH. for, if poſſible, let it be ſo to T a ſpace greater than the circle EFGH. therefore, inverſely, as the ſquare of FH to the ſquare of BD, ſo is





the ſpace T to the circle ABCD. but as the ſpace † T is to the circle ABCD, ſo is the circle EFGH to ſome ſpace, which muſt be leſs [d] than the circle ABCD, becauſe the ſpace T is greater, by hypotheſis, than the circle EFGH. therefore as the ſquare of FH is to

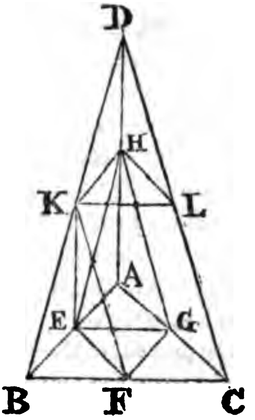
† For as in the foregoing Note at * it was explained how it was poſſible there could be a fourth proportional to the ſquares of BD, FH and the circle ABCD, which was named S. ſo in like manner there can be a fourth proportional to this other ſpace, named T, and the circles ABCD, EFGH, and the like is to be underſtood in ſome of the following Propoſitions.

the ſquare of BD, ſo is the circle EFGH to a ſpace leſs than the circle ABCD, which has been demonſtrated to be impoſſible. therefore the ſquare of BD is not to the ſquare of FH, as the circle ABCD is to any ſpace greater than the circle EFGH. and it has been demonſtrated that neither is the ſquare of BD to the ſquare of FH, as the circle ABCD to any ſpace leſs than the circle EFGH. wherefore as the ſquare of BD to the ſquare of FH, ſo is the circle ABCD to the circle EFGH ‡. Circles, therefore, are, &c. Q. E. D.

PROP. III. THEOR.

EVERY pyramid having a triangular baſe, may be divided into two equal and ſimilar pyramids having triangular baſes, and which are ſimilar to the whole pyramid; and into two equal priſms which together are greater than half of the whole pyramid. See N.

Let there be a pyramid of which the baſe is the triangle ABC and its vertex the point D. the pyramid ABCD may be divided into two equal and ſimilar pyramids having triangular baſes, and ſimilar to the whole; and into two equal priſms which together are greater than half of the whole pyramid.



Divide AB, BC, CA, AD, DB, DC, each into two equal parts in the points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Becauſe AE is equal to EB, and AH to HD, HE is parallel [a] to DB. for the ſame reaſon, HK is parallel to AB. therefore HEBK is a parallelogram, and HK equal [b] to EB. but EB is equal to AE; therefore alſo AE is equal to HK. and AH is equal to HD; wherefore EA, AH are equal to KH, HD, each to each; and the angle EAH is equal [c] to the angle KHD; therefore the baſe EH is

a. 2. 6.
b. 34. 1.
c. 29. 1.

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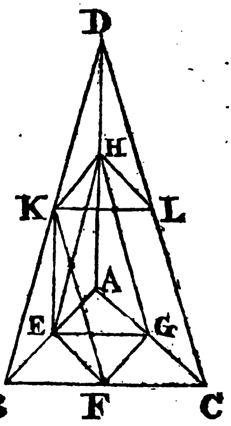
‡ Becauſe as a fourth proportional to the ſquares of BD, FH and the circle ABCD is poſſible, and that it can neither be leſs nor greater than the circle EFGH, it muſt be equal to it.



equal to the baſe KD, and the triangle AEH equal [d] and ſimilar to
the triangle HKD. for the ſame reaſon, the triangle AGH is equal
d. 4. 1. and ſimilar to the triangle HLD. and becauſe the two ſtraight lines
EH, HG which meet one another are parallel to KD, DL that
meet one another, and are not in the ſame plane with them, they
e. 10. 11. contain equal [e] angles; therefore the angle EHG is equal to the
angle KDL. again, becauſe EH, HG are equal to KD, DL, each
to each, and the angle EHG equal to the angle KDL; therefore
the baſe EG is equal to the baſe KL, and the triangle EHG equal
[d] and ſimilar to the triangle KDL. for the ſame reaſon, the triangle
AEG is alſo equal and ſimilar to the triangle HKL. Therefore
the pyramid of which the baſe is the triangle AEG, and of which
f. C. 11. the vertex is the point H, is equal [f] and
ſimilar to the pyramid the baſe of which
is the triangle KHL, and vertex the point
D. and becauſe HK is parallel to AB a ſide
of the triangle ADB, the triangle ADB is
equiangular to the triangle HDK, and
g. 4. 6. their ſides are proportionals [g]. therefore
the triangle ADB is ſimilar to the triangle
HDK. and for the ſame reaſon, the triangle
DBC is ſimilar to the triangle DKL;
and the triangle ADC to the triangle HDL;
and alſo the triangle ABC to the triangle
AEG. but the triangle AEG is ſimilar to
the triangle HKL, as before was proved,
h. 21. 6. therefore the triangle ABC is ſimilar [h] to the triangle HKL. and the
pyramid of which the baſe is the triangle ABC, and vertex the point
i. B. 11. & D, is therefore ſimilar [i] to the pyramid of which the baſe is the tri-
11. Def. 11. angle HKL, and vertex the ſame point D. but the pyramid of which
the baſe is the triangle HKL, and vertex the point D, is ſimilar, as has
been proved, to the pyramid the baſe of which is the triangle AEG,
and vertex the point H. wherefore the pyramid the baſe of which
is the triangle ABC, and vertex the point D, is ſimilar to the pyramid
of which the baſe is the triangle AEG and vertex H. therefore
each of the pyramids AEGH, HKLD is ſimilar to the whole
pyramid ABCD. and becauſe BF is equal to FC, the parallelogram
k. 41. 1. EBFG is double [k] of the triangle GFC. but when there are two
priſms of the ſame altitude, of which one has a parallelogram for
its baſe, and the other a triangle that is half of the parallelogram,

theſe priſms are equal [l] to one another; therefore the priſm having the parallelogram EBFG for its baſe, and the ſtraight line KH oppoſite to it, is equal to the priſm having the triangle GFC for its baſe, and the triangle HKL oppoſite to it; for they are of the ſame altitude, becauſe they are between the parallel [m] planes ABC, HKL. and it is manifeſt that each of theſe priſms is greater than either of the pyramids of which the triangles AEG, HKL are the baſes, and the vertices the points H, D; becauſe if EF be joined, the priſm having the parallelogram EBFG for its baſe, and KH the ſtraight line oppoſite to it, is greater than the pyramid of which the baſe is the triangle EBF, and vertex the point K; but this pyramid is equal [f] to the pyramid the baſe of which is the triangle AEG, and vertex the point H; becauſe they are contained by equal and ſimilar planes. wherefore the priſm having the parallelogram EBFG for its baſe, and oppoſite ſide KH, is greater than the pyramid of which the baſe is the triangle AEG, and vertex the point H. and the priſm of which the baſe is the parallelogram EBFG, and oppoſite ſide KH is equal to the priſm having the triangle GFC for its baſe, and HKL the triangle oppoſite to it; and the pyramid of which the baſe is the triangle AEG, and vertex H, is equal to the pyramid of which the baſe is the triangle HKL, and vertex D. therefore the two priſms before-mentioned are greater than the two pyramids of which the baſes are the triangles AEG, HKL, and vertices the points H, D. Therefore the whole pyramid of which the baſe is the triangle ABC, and vertex the point D, is divided into two equal pyramids ſimilar to one another, and to the whole pyramid; and into two equal priſms; and the two priſms are together greater than half of the whole pyramid. Q. E. D.

l. 40. 11.

m. 15. 11.

f. C. 11.

PROP. IV. THEOR.

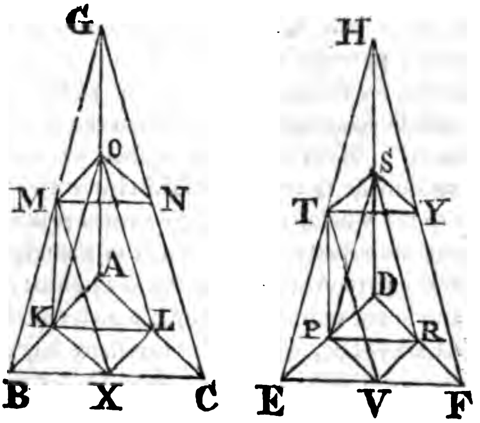
See N. IF there be two pyramids of the ſame altitude, upon triangular baſes, and each of them be divided into two equal pyramids ſimilar to the whole pyramid, and alſo into two equal priſms; and if each of theſe pyramids be divided in the ſame manner as the firſt two, and ſo on. as the baſe of one of the firſt two pyramids is to the baſe of the other, ſo ſhall all the priſms in one of them be to all the priſms in the other, that are produced by the ſame number of diviſions.

Let there be two pyramids of the ſame altitude upon the triangular baſes ABC, DEF, and having their vertices in the points G, H; and let each of them be divided into two equal pyramids ſimilar to the whole, and into two equal priſms; and let each of the pyramids thus made be conceived to be divided in the like manner, and ſo on. as the baſe ABC is to the baſe DEF, ſo are all the priſms in the pyramid ABCG to all the priſms in the pyramid DEFH made by the ſame number of diviſions.

Make the ſame conſtruction as in the foregoing propoſition. and becauſe BX is equal to XC, and AL to LC, therefore XL is parallel [a] to AB, and the triangle ABC ſimilar to the triangle LXC. for the ſame reaſon, the triangle DEF is ſimilar to RVF. and becauſe BC is double of CX, and EF double of FV, therefore BC is to CX, as EF to FV. and upon BC, CX are deſcribed the ſimilar and ſimilarly ſituated rectilineal figures ABC, LXC; and upon EF, FV, in like manner, are deſcribed the ſimilar figures DEF, RVF. therefore as the triangle ABC is to the triangle LXC, ſo [b] is the triangle DEF to the triangle RVF, and, by permutation, as the triangle ABC to the triangle DEF, ſo is the triangle LXC to the triangle RVF. and becauſe the planes ABC, OMN, as alſo the planes DEF, STY are parallel [c], the perpendiculars drawn from the points G, H to the baſes ABC, DEF, which, by the Hypotheſis, are equal to one another, ſhall be cut each into two equal [d] parts by the planes OMN, STY, becauſe the ſtraight lines GC, HF are cut into two equal parts in the points N, Y by the ſame planes. therefore the priſms LXCOMN, RVFSTY are of the ſame altitude; and

a. 2. 6.
b. 22. 6.
c. 15. 11.
d. 17. 11.

therefore as the baſe LXC to the baſe RVF; that is, as the triangle ABC to the triangle DEF, ſo [e] is the priſm having the triangle LXC for its baſe, and OMN the triangle oppoſite to it, to the priſm of which the baſe is the triangle RVF, and the oppoſite triangle STY. and becauſe the two priſms in the pyramid ABCG are equal to one another, and alſo the two priſms in the pyramid DEFH equal to one another, as the priſm of which the baſe is the parallelogram KBXL and oppoſite ſide MO, to the priſm having the triangle LXC for its baſe, and OMN the triangle oppoſite to it; ſo [f] is the priſm of which the baſe is the parallelogram PEVR, and oppoſite ſide TS, to the priſm of which the baſe is the triangle RVF, and oppoſite triangle STY. therefore, componendo, as the priſms KBXLMO, LXCOMN together are unto the priſm LXCOMN; ſo are the priſms PEVRTS, RVFSTY to the priſm RVFSTY. and, permutando, as the priſms KBXLMO, LXCOMN are to the priſms PEVRTS, RVFSTY; ſo is the priſm LXCOMN to the priſm RVFSTY. but as the priſm LXCOMN to the priſm RVFSTY, ſo is, as has been proved, the baſe ABC to the baſe DEF. therefore as the baſe ABC to the baſe DEF, ſo are the two priſms in the pyramid ABCG to the two priſms in the pyramid DEFH. and likewiſe if the pyramids now made, for example the two OMNG, STYH be divided in the ſame manner; as the baſe OMN is to the baſe STY, ſo ſhall the two priſms in the pyramid OMNG be to the two priſms in the pyramid STYH. but the baſe OMN is to the baſe STY, as the baſe ABC to the baſe DEF; therefore as the baſe



[e] Cor. 32. 11.

[f] 7. 5.

ABC to the baſe DEF, ſo are the two priſms in the pyramid ABCG to the two priſms in the pyramid DEFH; and ſo are the two priſms in the pyramid OMNG to the two priſms in the pyramid STYH; and ſo are all four to all four. and the ſame thing may be ſhewn of the priſms made by dividing the pyramids AKLO and DPRS, and of all made by the ſame number of diviſions. Q. E. D.

PROP. V. THEOR.

See N. PYRAMIDS of the ſame altitude which have triangular baſes, are to one another as their baſes.

Let the pyramids of which the triangles ABC, DEF are the baſes, and of which the vertices are the points G, H, be of the ſame altitude. as the baſe ABC to the baſe DEF, ſo is the pyramid ABCG to the pyramid DEFH.

For, if it be not ſo, the baſe ABC muſt be to the baſe DEF, as the pyramid ABCG to a ſolid either leſs than the pyramid DEFH, or greater than it *. Firſt, let it be to a ſolid leſs than it, viz. to the ſolid Q. and divide the pyramid DEFH into two equal pyramids, ſimilar to the whole, and into two equal priſms. therefore a. 3. 12. theſe two priſms are greater [a] than the half of the whole pyramid. and, again, let the pyramids made by this diviſion be in like manner divided, and ſo on, until the pyramids which remain undivided in the pyramid DEFH be, all of them together, leſs than the exceſs of the pyramid DEFH above the ſolid Q. let theſe, for example, be the pyramids DPRS, STYH. therefore the priſms, which make the reſt of the pyramid DEFH, are greater than the ſolid Q. divide likewiſe the pyramid ABCG in the ſame manner, and into as many parts, as the pyramid DEFH. therefore as the baſe ABC b. 4. 12. to the baſe DEF, ſo [b] are the priſms in the pyramid ABCG to the priſms in the pyramid DEFH. but as the baſe ABC to the baſe DEF, ſo, by hypotheſis, is the pyramid ABCG to the ſolid Q; and therefore, as the pyramid ABCG to the ſolid Q, ſo are the priſms in the pyramid ABCG to the priſms in the pyramid DEFH. but the pyramid ABCG is greater than the priſms contained in it; wherefore [c] c. 14. 5. alſo the ſolid Q is greater than the priſms in the pyramid DEFH. but it is alſo leſs, which is impoſſible.

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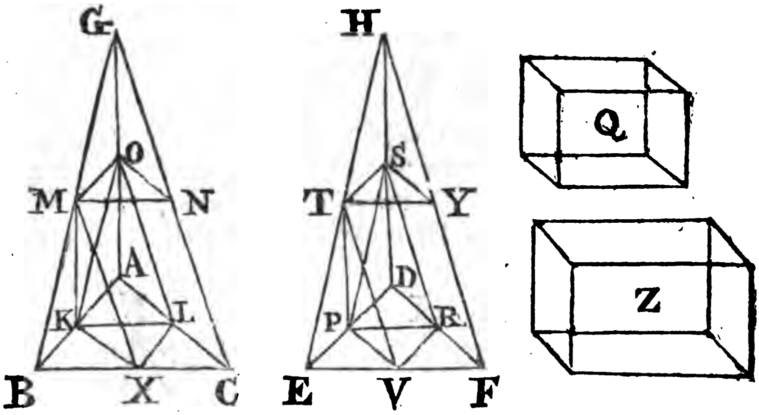
* This may be explained the ſame way as at the note * in Propoſition 2. in the like caſe.

therefore the baſe ABC is not to the baſe DEF, as the pyramid ABCG to any ſolid which is leſs than the pyramid DEFH. in the ſame manner it may be demonſtrated, that the baſe DEF is not to the baſe ABC, as the pyramid DEFH to any ſolid which is leſs than the pyramid ABCG. Nor can the baſe ABC be to the baſe DEF, as the pyramid ABCG to any ſolid which is greater than the pyramid DEFH. for, if it be poſſible, let it be ſo to a greater, viz. the ſolid Z. and becauſe the baſe ABC is to the baſe DEF, as the pyramid ABCG to the ſolid Z; by inverſion, as the baſe DEF to the baſe ABC, ſo is the ſolid Z to the pyramid ABCG. but as the ſolid Z is to the pyramid ABCG, ſo is the pyramid



DEFH to ſome ſolid †, which muſt be leſs [d] than the pyramid ABCG, becauſe the ſolid Z is greater than the pyramid DEFH. and therefore, as the baſe DEF to the baſe ABC, ſo is the pyramid DEFH to a ſolid leſs than the pyramid ABCG; the contrary to which has been proved. therefore the baſe ABC is not to the baſe DEF, as the pyramid ABCG to any ſolid which is greater than the pyramid DEFH. and it has been proved that neither is the baſe ABC to the baſe DEF, as the pyramid ABCG to any ſolid which is leſs than the pyramid DEFH. Therefore as the baſe ABC is to the baſe DEF, ſo is the pyramid ABCG to the pyramid DEFH. Wherefore pyramids, &c. Q. E. D.

d. 14. 5.

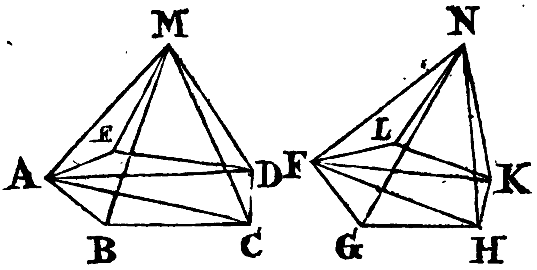
† This may be explained the ſame way as the like at the mark † in Prop. 2.

PROP. VI. THEOR.

See N. PYRAMIDS of the ſame altitude which have polygons for their baſes, are to one another as their baſes.

Let the pyramids which have the polygons ABCDE, FGHKL for their baſes, and their vertices in the points M, N, be of the ſame altitude. as the baſe ABCDE to the baſe FGHKL, ſo is the pyramid ABCDEM to the pyramid FGHKLN.

Divide the baſe ABCDE into the triangles ABC, ACD, ADE; and the baſe FGHKL into the triangles FGH, FHK, FKL. and upon the baſes ABC, ACD, ADE let there be as many pyramids of which the common vertex is the point M. and upon the remaining baſes as many pyramids having their common vertex in the point
a. 5. 12. N. therefore, ſince the triangle ABC is to the triangle FGH, as [a] the pyramid ABCM to the pyramid FGHN; and the triangle ACD to the triangle FGH, as the pyramid ACDM to the pyramid FGHN; and alſo the triangle ADE to the triangle FGH, as the pyramid



ADEM to the pyramid FGHN; as all the firſt antecedents to their
b. 2. Cor. 24. 5. common conſequent, ſo [b] are all the other antecedents to their common conſequent; that is, as the baſe ABCDE to the baſe FGH, ſo is the pyramid ABCDEM to the pyramid FGHN. and for the ſame reaſon, as the baſe FGHKL to the baſe FGH, ſo is the pyramid FGHKLN to the pyramid FGHN. and, by inverſion, as the baſe FGH to the baſe FGHKL, ſo is the pyramid FGHN to the pyramid FGHKLN. then becauſe as the baſe ABCDE to the baſe FGH, ſo is the pyramid ABCDEM to the pyramid FGHN; and as the baſe FGH to the baſe FGHKL, ſo is the pyramid FGHN to the
c. 22. 5. pyramid FGHKLN; therefore, ex aequali [c], as the baſe ABCDE to

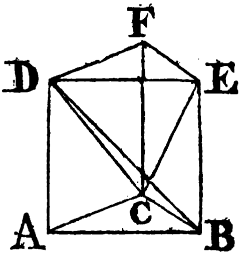
the baſe FGHKL, ſo the pyramid ABCDEM to the pyramid FGHKLN. Therefore pyramids, &c. Q. E. D.

PROP. VII. THEOR.

EVERY priſm having a triangular baſe may be divided into three pyramids that have triangular baſes, and are equal to one another.

Let there be a priſm of which the baſe is the triangle ABC, and let DEF be the triangle oppoſite to it. the priſm ABCDEF may be divided into three equal pyramids having triangular baſes.

Join BD, EC, CD; and becauſe ABED is a parallelogram of which BD is the diameter, the triangle ABD is equal[a] to the triangle EBD; therefore the pyramid of which the baſe is the triangle ABD, and vertex the point C, is equal[b] to the pyramid of which the baſe is the triangle EBD, and vertex the point C. but this pyramid is the ſame with the pyramid the baſe of which is the triangle EBC, and vertex the point D; for they are contained by the ſame planes. therefore the pyramid of which the baſe is the triangle ABD, and vertex the point C, is equal to the pyramid the baſe of which is the triangle EBC, and vertex the point D. again, becauſe FCBE is a parallelogram of which the diameter is CE, the triangle ECF is equal[a] to the triangle ECB; therefore the pyramid of which the baſe is the triangle ECB, and vertex the point D, is equal to the pyramid the baſe of which is the triangle ECF, and vertex the point D. but the pyramid of which the baſe is the triangle ECB, and vertex the point D has been proved equal to the pyramid of which the baſe is the triangle ABD, and vertex the point C. Therefore the priſm ABCDEF is divided into three equal pyramids having triangular baſes, viz. into the pyramids ABDC, EBDC, ECFD. and becauſe the pyramid of which the baſe is the triangle ABD, and vertex the point C, is the ſame with the pyramid of which the baſe is the triangle ABC, and vertex the point D, for they are contained by the ſame planes; and that the pyramid of which the baſe is the triangle ABD, and vertex the point C, has been demonſtrated to be a third part of the priſm the



a. 34. 1.

b. 5. 12.

baſe of which is the triangle ABC, and to which DEF is the oppoſite triangle; therefore the pyramid of which the baſe is the triangle ABC, and vertex the point D, is the third part of the priſm which has the ſame baſe, viz. the triangle ABC, and DEF is the oppoſite triangle. Q. E. D.

Cor. 1. From this it is manifeſt, that every pyramid is the third part of a priſm which has the ſame baſe, and is of an equal altitude with it; for if the baſe of the priſm be any other figure than a triangle, it may be divided into priſms having triangular baſes.

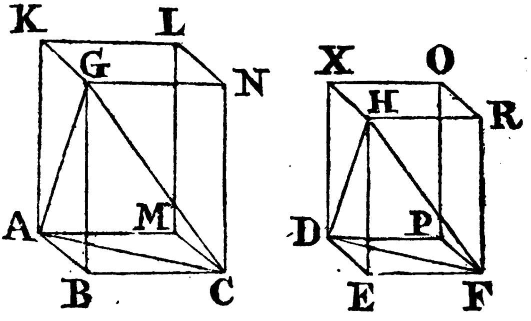
Cor. 2. Priſms of equal altitudes are to one another as their baſes; becauſe the pyramids upon the ſame baſes, and of the ſame
c. 6. 12. altitude, are [c] to one another as their baſes.

PROP. VIII. THEOR.

SIMILAR pyramids having triangular baſes, are one to another in the triplicate ratio of that of their homologous ſides.

Let the pyramids having the triangles ABC, DEF for their baſes, and the points G, H for their vertices, be ſimilar and ſimilarly ſituated. the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which the ſide BC has to the homologous ſide EF.

Complete the parallelograms ABCM, GBCN, ABGK, and the ſolid parallelepiped BGML contained by theſe planes and thoſe oppoſite to them. and in like manner complete the ſolid parallelepiped EHPO contained by the three parallelograms DEFP, HEFR,
a. 11. Def. 11. DEHX, and thoſe oppoſite to them. and becauſe the pyramid ABCG is ſimilar to the pyramid DEFH, the angle ABC is equal [a]



to the angle DEF, and the angle GBC to the angle HEF, and ABG
to DEH. and AB is [b] to BC, as DE to EF; that is, the ſides about
the equal angles are proportionals; wherefore the parallelogram b. 1. Def. 6.
BM is ſimilar to EP. for the ſame reaſon, the parallelogram BN is
ſimilar to ER, and BK to EX. therefore the three parallelograms
BM, BN, BK are ſimilar to the three EP, ER, EX. but the three
BM, BN, BK are equal and ſimilar [c] to the three which are oppoſite c. 24. 11.
to them, and the three EP, ER, EX equal and ſimilar to the
three oppoſite to them. wherefore the ſolids BGML, EHPO are
contained by the ſame number of ſimilar planes; and their ſolid
angles are equal [d]; and therefore the ſolid BGML is ſimilar [a] to d. B. 11.
the ſolid EHPO. but ſimilar ſolid parallelepipeds have the triplicate
[e] ratio of that which their homologous ſides have. therefore the e. 33. 11.
ſolid BGML has to the ſolid EHPO the triplicate ratio of that
which the ſide BC has to the homologous ſide EF. but as the ſolid
BGML is to the ſolid EHPO, ſo is [f] the pyramid ABCG to the f. 15. 5.
pyramid DEFH; becauſe the pyramids are the ſixth part of the
ſolids, ſince the priſm, which is the half [g] of the ſolid parallelepiped, g. 28. 11.
is triple [h] of the pyramid. Wherefore likewiſe the pyramid ABCG h. 7. 12.
has to the pyramid DEFH, the triplicate ratio of that which BC
has to the homologous ſide EF. Q. E. D.

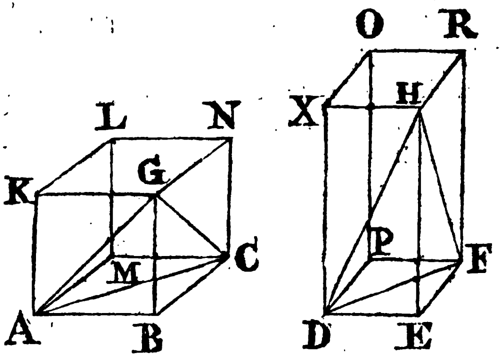
Cor. From this it is evident, that ſimilar pyramids which have See N.
multangular baſes, are likewiſe to one another in the triplicate ratio
of their homologous ſides. for, they may be divided into ſimilar
pyramids having triangular baſes, becauſe the ſimilar polygons
which are their baſes may be divided into the ſame number of
ſimilar triangles homologous to the whole polygons; therefore as
one of the triangular pyramids in the firſt multangular pyramid is
to one of the triangular pyramids in the other, ſo are all the triangular
pyramids in the firſt to all the triangular pyramids in the
other; that is, ſo is the firſt multangular pyramid to the other.
but one triangular pyramid is to its ſimilar triangular pyramid, in
the triplicate ratio of their homologous ſides; and therefore the
firſt multangular pyramid has to the other, the triplicate ratio of
that which one of the ſides of the firſt has to the homologous
ſide of the other.

PROP. IX. THEOR.

THE baſes and altitudes of equal pyramids having triangular baſes are reciprocally proportional. and triangular pyramids of which the baſes and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids of which the triangles ABC, DEF are the baſes, and which have their vertices in the points G, H be equal to one another. the baſes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional, viz. the baſe ABC is to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, HF; and the ſolid parallelepipeds BGML, EHPO contained by theſe planes and thoſe oppoſite to them. and becauſe the pyramid ABCG is equal to the pyramid DEFH, and that the ſolid BGML is ſextuple of the pyramid ABCG, and the ſolid EHPO ſextuple of the pyramid DEFH; a. 1. Ax. 5. therefore the ſolid BGML is equal [a] to the ſolid EHPO. but the baſes and altitudes of equal ſolid parallelepipeds are b. 34. 11. reciprocally proportional [b]; therefore as the baſe BM to the baſe EP, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid c. 15. 5. BGML. but as the baſe BM to the baſe EP, ſo is [c] the triangle ABC to the triangle DEF; therefore as the triangle ABC to the triangle DEF, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid BGML. but the altitude of the ſolid EHPO is the ſame with the altitude of the pyramid DEFH; and the altitude of



the ſolid BGML is the ſame with the altitude of the pyramid ABCG. therefore, as the baſe ABC to the baſe DEF, ſo is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG. wherefore the baſes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional.

Again, Let the baſes and altitudes of the pyramids ABCG, DEFH be reciprocally proportional, viz. the baſe ABC to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG. the pyramid ABCG is equal to the pyramid DEFH.

The ſame conſtruction being made, becauſe as the baſe ABC to the baſe DEF, ſo is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG; and as the baſe ABC to the baſe DEF, ſo is the parallelogram BM to the parallelogram EP; therefore the parallelogram BM is to EP, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG. but the altitude of the pyramid DEFH is the ſame with the altitude of the ſolid parallelepiped EHPO; and the altitude of the pyramid ABCG is the ſame with the altitude of the ſolid parallelepiped BGML. as, therefore, the baſe BM to the baſe EP, ſo is the altitude of the ſolid parallelepiped EHPO to the altitude of the ſolid parallelepiped BGML. but ſolid parallelepipeds having their baſes and altitudes reciprocally proportional, are equal[b] to one another. therefore the ſolid parallelepiped BGML is equal to the ſolid parallelepiped EHPO. and the pyramid ABCG is the ſixth part of the ſolid BGML, and the pyramid DEFH the ſixth part of the ſolid EHPO. therefore the pyramid ABCG is equal to the pyramid DEFH. Therefore the baſes, &c. Q. E. D.

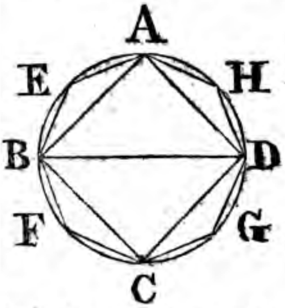
[b] 34. 11.

PROP. X. THEOR.

EVERY cone is the third part of a cylinder which has the ſame baſe, and is of an equal altitude with it.

Let a cone have the ſame baſe with a cylinder, viz. the circle ABCD, and the ſame altitude. the cone is the third part of the cylinder; that is, the cylinder is triple of the cone.

If the cylinder be not triple of the cone, it muſt either be greater than the triple, or leſs than it. Firſt, Let it be greater than the triple; and deſcribe the ſquare ABCD in the circle; this ſquare is greater than the half of the circle ABCD †. upon the ſquare ABCD erect a priſm of the ſame altitude with the cylinder; this priſm is greater than half of the cylinder; becauſe if a ſquare be deſcribed about the circle, and a priſm erected upon the ſquare, of the ſame altitude with the cylinder, the inſcribed ſquare is half of that circumſcribed; and upon theſe ſquare baſes are erected ſolid parallelepipeds, viz. the priſms, of the ſame altitude; therefore the priſm upon the ſquare ABCD is the half of the priſm upon the ſquare deſcribed about the circle; becauſe they are to one another as their a. 32. 11. baſes [a]. and the cylinder is leſs than the priſm upon the ſquare deſcribed about the circle ABCD. therefore the priſm upon the ſquare ABCD of the ſame altitude with the cylinder, is greater than half of the cylinder. Biſect the circumferences AB, BC, CD, DA in the points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA. then, each of the triangles AEB, BFC, CGD, DHA is greater than the half of the ſegment of the circle in which it ſtands, as was ſhewn in Prop. 2. of this Book. Erect priſms upon each of theſe triangles of the ſame altitude with the cylinder; each of theſe priſms is greater than half of the ſegment of the cylinder in which it is; becauſe if thro' the points E, F, G, H parallels be drawn to AB, BC, CD, DA, and parallelograms be completed upon the ſame AB, BC, CD, DA, and ſolid parallelepipeds be erected upon the parallelograms; the priſms upon the triangles AEB, BFC, CGD, b. 2. Cor. 7. 12. DHA are the halves of the ſolid parallelepipeds [b]. and the ſegments of the cylinder which are upon the ſegments of the circle cut off by AB, BC, CD, DA, are leſs than the ſolid parallelepipeds which contain them. therefore the priſms upon the triangles AEB, BFC, CGD, DHA, are greater than half of the ſegments of the cylinder in which they are. therefore if each of the circumferences be divided into two equal parts, and ſtraight lines be drawn from the points of diviſion to the extremities of the circumferences, and upon the triangles thus made priſms be erected of the ſame altitude with the cylinder, and ſo on, there muſt at length remain ſome c. Lemma. ſegments of the cylinder which together are leſs [c] than the exceſs of the cylinder above the triple of the cone. let them be thoſe upon the



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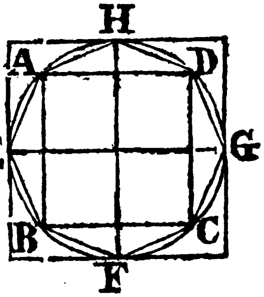
† As was ſhewn in Prop. 2. of this Book.

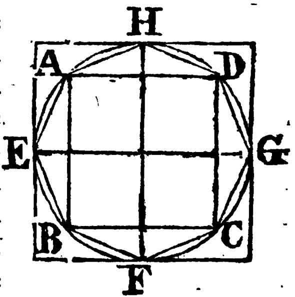
ſegments of the circle AE, EB, BF, FC, CG, GD, DH, HA. therefore the reſt of the cylinder, that is the priſm of which the baſe is the polygon AEBFCGDH, and of which the altitude is the ſame with that of the cylinder, is greater than the triple of the cone. but this priſm is triple [d] of the pyramid upon the ſame baſe, of which the vertex is the ſame with the vertex of the cone; therefore the pyramid upon the baſe AEBFCGDH, having the ſame vertex with the cone, is greater than the cone, of which the baſe is the circle ABCD. but it is alſo leſs, for the pyramid is contained within the cone; which is impoſſible. Nor can the cylinder be leſs than the triple of the cone. let it be leſs if poſſible. therefore, inverſely, the cone is greater than the third part of the cylinder. In the circle ABCD deſcribe a ſquare, this ſquare is greater than the half of the circle. and upon the ſquare ABCD erect a pyramid having the ſame vertex with the cone; this pyramid is greater than the half of the cone; becauſe, as was before demonſtrated, if a ſquare be deſcribed about the circle, the ſquare ABCD is the half of it; and if upon theſe ſquares there be erected ſolid parallelepipeds of the ſame altitude with the cone, which are alſo priſms, the priſm upon the ſquare ABCD ſhall be the half of that which is upon the ſquare deſcribed about the circle; for they are to one another as their baſes [a]; as are alſo the third parts of them. therefore the pyramid the baſe of which is the ſquare ABCD is half of the pyramid upon the ſquare deſcribed about the circle. but this laſt pyramid is greater than the cone which it contains; therefore the pyramid upon the ſquare ABCD having the ſame vertex with the cone, is greater than the half of the cone. Biſect the circumferences AB, BC, CD, DA in the points E, F, G, H, and join AE, EB, BF, FC, CG, GD, DH, HA. therefore each of the triangles AEB, BFC, CGD, DHA is greater than half of the ſegment of the circle in which it is. upon each of theſe triangles erect pyramids having the ſame vertex with the cone. therefore each of theſe pyramids is greater than the half of the ſegment of the cone in which it is, as before was demonſtrated of the priſms and ſegments of the cylinder. and thus dividing each of the circumferences into two equal parts, and joining the points of diviſion and their extremi-



d. 1. Cor. 7. 12.

a. 32. 11.

ties by ſtraight lines, and upon the triangles erecting pyramids having their vertices the ſame with that of the cone, and ſo on, there muſt at length remain ſome ſegments of the cone which together ſhall be leſs than the exceſs of the cone above the third part of the cylinder. let theſe be the ſegments upon AE, EB, BF, FC, CG, GD, DH, HA. therefore the reſt of the cone, that is the pyramid, of which the baſe is the polygon AEBFCGDH, and of which the vertex is the ſame with that of the cone, is greater than the third part of the cylinder. but this pyramid is the third part of the priſm upon the ſame baſe AEBFCGDH, and of the ſame altitude with the cylinder. therefore this priſm is greater than the cylinder of which the baſe is the circle ABCD. but it is alſo leſs, for it is contained within the cylinder; which is impoſſible. therefore the cylinder is not leſs than the triple of the cone. and it has been demonſtrated that neither is it greater than the triple. therefore the cylinder is triple of the cone, or, the cone is the third part of the cylinder. Wherefore every cone, &c. Q. E. D.



PROP. XI. THEOR.

See N.

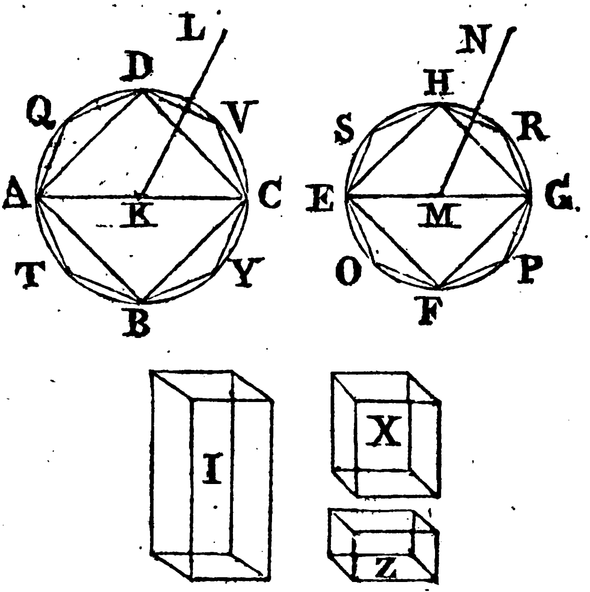
CONES and cylinders of the ſame altitude, are to one another as their baſes.

Let the cones and cylinders, of which the baſes are the circles ABCD, EFGH, and the axes KL, MN, and AC, EG the diameters of their baſes, be of the ſame altitude. as the circle ABCD to the circle EFGH, ſo is the cone AL to the cone EN.

If it be not ſo, let the circle ABCD be to the circle EFGH, as the cone AL to ſome ſolid either leſs than the cone EN, or greater than it. Firſt, let it be to a ſolid leſs than EN, viz. to the ſolid X; and let Z be the ſolid which is equal to the exceſs of the cone EN above the ſolid X; therefore the cone EN is equal to the ſolids X, Z together. in the circle EFGH deſcribe the ſquare EFGH, therefore this ſquare is greater than the half of the circle. upon the ſquare EFGH erect a pyramid of the ſame altitude with the cone; this pyramid is greater than half of the cone. for if a ſquare be deſcribed about the circle, and a pyramid be erected upon it, having the ſame vertex with the cone†, the pyramid inſcribed in the cone is half of the pyramid circumſcribed about it, becauſe they are to one another as their baſes[a]. but the cone is leſs than the circumſcribed pyramid; therefore the pyramid of which the baſe is the ſquare EFGH, and its vertex the ſame with that of the cone, is greater than half of the cone. divide the circumferences EF, FG, GH, HE, each into two equal parts in the points O, P, R, S, and join EO, OF, FP, PG, GR, RH, HS, SE. therefore each of the triangles EOF, FPG, GRH, HSE is greater than half of the ſegment of the circle in which it is. upon each of theſe triangles erect a pyramid having the ſame vertex with the cone; each of theſe pyramids is greater than the half of the ſegment of the cone in which it is. and thus dividing each of theſe circumferences into two equal parts, and from the points of diviſion drawing ſtraight lines to the extremities of the circumferences, and upon each of the triangles thus made erecting pyramids having the ſame vertex with the cone, and ſo on, there muſt at length remain ſome ſegments of the cone which are together leſs[b] than the ſolid Z. let theſe be the ſegments upon EO, OF, FP, PG,

a. 6. 12.

b. Lemma.



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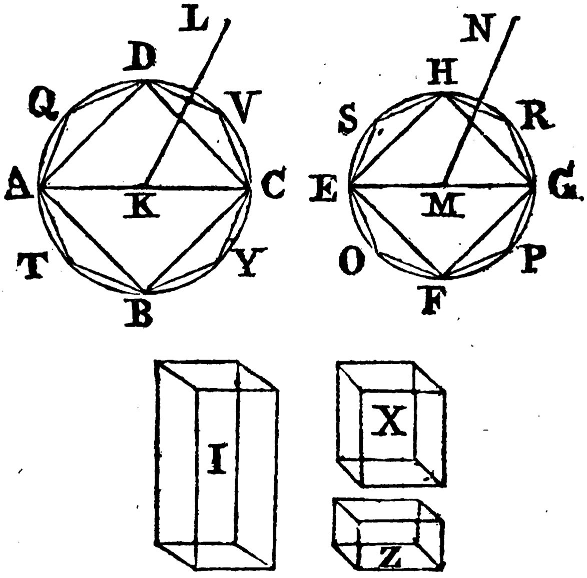
† Vertex is put in place of altitude which is in the Greek, becauſe the pyramid, in what follows, is ſuppoſed to be circumſcribed about the cone, and ſo muſt have the ſame vertex. and the ſame change is made in ſome places following.

GR, RH, HS, SE. therefore the remainder of the cone, viz. the
pyramid of which the baſe is the polygon EOFPGRHS, and its
vertex the ſame with that of the cone, is greater than the ſolid X.
In the circle ABCD deſcribe the polygon ATBYCVDQ ſimilar to
the polygon EOFPGRHS, and upon it erect a pyramid of the ſame
altitude with the cone AL. and becauſe as the ſquare of AC is to
c. 1. 12. the ſquare of EG, ſo [c] is the polygon ATBYCVDQ to the polygon
EOFPGRHS; and as the ſquare of AC to the ſquare of EG, ſo
d. 2. 12. is [d] the circle ABCD to the circle EFGH; therefore the circle
e. 11. 5. ABCD is [e] to the circle EFGH, as the polygon ATBYCVDQ to



the polygon EOFPGRHS. but as the circle ABCD to the circle
EFGH, ſo is the cone AL to the ſolid X; and as the polygon
a. 6. 12. ATBYCVDQ to the polygon EOFPGRHS, ſo is [a] the pyramid of
which the baſe is the firſt of thoſe polygons, and vertex L, to the
pyramid of which the baſe is the other polygon, and its vertex N.
therefore as the cone AL to the ſolid X, ſo is the pyramid of which
the baſe is the polygon ATBYCVDQ, and vertex L to the pyramid
the baſe of which is the polygon EOFPGRHS, and vertex N. but
the cone AL is greater than the pyramid contained in it; therefore
f. 14. 5. the ſolid X is greater [f] than the pyramid in the cone EN. but it is
leſs, as was ſhewn; which is abſurd. therefore the circle ABCD is

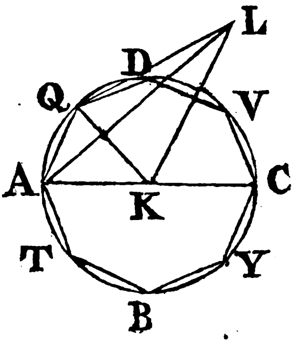
not to the circle EFGH, as the cone AL to any ſolid which is leſs than the cone EN. In the ſame manner it may be demonſtrated that the circle EFGH is not to the circle ABCD, as the cone EN to any ſolid leſs than the cone AL. Nor can the circle ABCD be to the circle EFGH, as the cone AL to any ſolid greater than the cone EN. for, if it be poſſible, let it be ſo to the ſolid I which is greater than the cone EN. therefore, by inverſion, as the circle EFGH to the circle ABCD, ſo is the ſolid I to the cone AL. but as the ſolid I to the cone AL, ſo is the cone EN to ſome ſolid, which muſt be leſs [f] than the cone AL, becauſe the ſolid I is greater f. 14. 5. than the cone EN. therefore as the circle EFGH is to the circle ABCD, ſo is the cone EN to a ſolid leſs than the cone AL, which was ſhewn to be impoſſible. therefore the circle ABCD is not to the circle EFGH, as the cone AL is to any ſolid greater than the cone EN. and it has been demonſtrated that neither is the circle ABCD to the circle EFGH, as the cone AL to any ſolid leſs than the cone EN. therefore the circle ABCD is to the circle EFGH, as the cone AL to the cone EN. but as the cone is to the cone, ſo [g] is the cylinder to the cylinder; becauſe the cylinders are triple g. 15. 5. [h] of the cones, each of each. Therefore as the circle ABCD to h. 10. 12. the circle EFGH, ſo are the cylinders upon them of the ſame altitude. Wherefore cones and cylinders of the ſame altitude, are to one another as their baſes. Q. E. D.

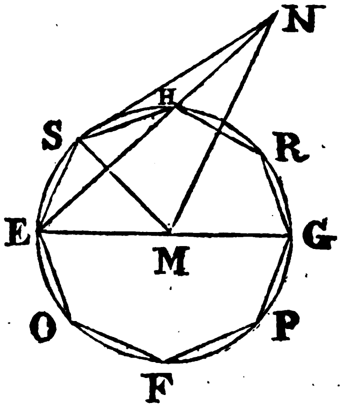
PROP. XII. THEOR.

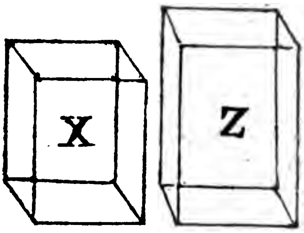
SIMILAR cones and cylinders have to one another the triplicate ratio of that which the diameters of their baſes have. See N.

Let the cones and cylinders of which the baſes are the circles ABCD, EFGH, and the diameters of the baſes AC, EG, and KL, MN the axes of the cones or cylinders, be ſimilar. the cone of which the baſe is the circle ABCD, and vertex the point L, has to the cone of which the baſe is the circle EFGH, and vertex N, the triplicate ratio of that which AC has to EG.

For if the cone ABCDL has not to the cone EFGHN the triplicate ratio of that which AC has to EG, the cone ABCDL ſhall have the triplicate of that ratio to ſome ſolid which is leſs or greater than the cone EFGHN. Firſt, let it have it to a leſs, viz. to the ſolid X. make the ſame conſtruction as in the preceding Propoſition, and it may be demonſtrated the very ſame way as in that Propoſition, that the pyramid of which the baſe is the polygon EOFPGRHS, and vertex N is greater than the ſolid X. Deſcribe alſo in the circle ABCD the polygon ATBYCVDQ ſimilar to the polygon EOFPGRHS, upon which erect a pyramid having the ſame vertex with the cone; and let LAQ be one of the triangles containing the pyramid upon the polygon ATBYCVDQ the vertex of which is



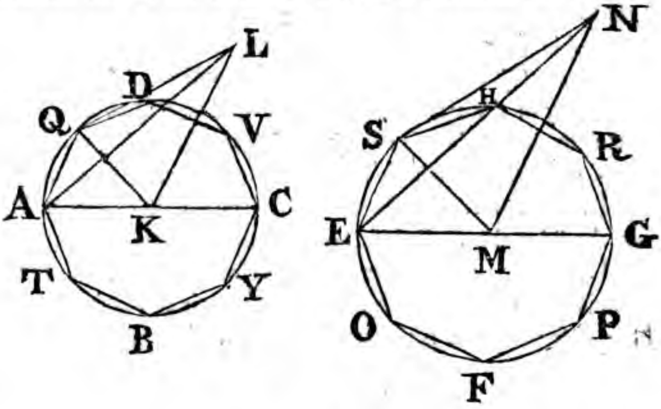


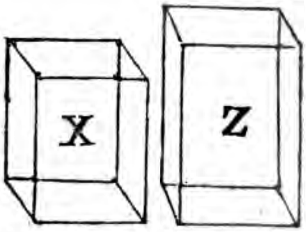


L; and let NES be one of the triangles containing the pyramid upon the polygon EOFPGRHS of which the vertex is N; and join KQ, MS. becauſe then the cone ABCDL is ſimilar to the cone EFGHN,
a. 24. Def. 11. AC is [a] to EG, as the axis KL to the axis MN; and as AC to EG,
b. 15. 5. ſo [b] is AK to EM; therefore as AK to EM, ſo is KL to MN; and, alternately, AK to KL, as EM to MN. and the right angles AKL, EMN are equal; therefore, the ſides about theſe equal angles being
c. 6. 6. proportionals, the triangle AKL is ſimilar [c] to the triangle EMN. again, becauſe AK is to KQ, as EM to MS, and that theſe ſides

are about equal angles AKQ, EMS, becauſe theſe angles are, each of them, the ſame part of four right angles at the centers K, M; therefore the triangle AKQ is ſimilar [c] to the triangle EMS. c. 6. 6. and becauſe it has been ſhewn that as AK to KL, ſo is EM to MN, and that AK is equal to KQ, and EM to MS, as QK to KL, ſo is SM to MN; and therefore, the ſides about the right angles QKL, SMN being proportionals, the triangle LKQ is ſimilar to the triangle NMS. and becauſe of the ſimilarity of the triangles AKL, EMN, as LA is to AK, ſo is NE to EM; and by the ſimilarity of the triangles AKQ, EMS, as KA to AQ, ſo ME to ES; ex aequali [d], d. 22. 5. LA is to AQ, as NE to ES. again, becauſe of the ſimilarity of the triangles LQK, NSM, as LQ to QK, ſo NS to SM; and from the ſimilarity of the triangles KAQ, MES, as KQ to QA, ſo MS to SE; ex aequali [d], LQ is to QA, as NS to SE. and it was proved that QA is to AL, as SE to EN; therefore, again, ex aequali, as QL to LA, ſo is SN to NE. wherefore the triangles LQA, NSE, having the ſides about all their angles proportionals, are equiangular [e] and ſimilar to one another. and therefore the pyramid of which e. 5. 6. the baſe is the triangle AKQ, and vertex L, is ſimilar to the pyramid the baſe of which is the triangle EMS, and vertex N, becauſe their ſolid angles are equal [f] to one another, and they are contained f. B. 11. by the ſame number of ſimilar planes. but ſimilar pyramids which have triangular baſes have to one another the triplicate [g] ratio of that g. 8. 12. which their homologous ſides have; therefore the pyramid AKQL has to the pyramid EMSN the triplicate ratio of that which AK has to EM. In the ſame manner, if ſtraight lines be drawn from the points D, V, C, Y, B, T to K, and from the points H, R, G, P, F, O to M, and pyramids be erected upon the triangles having the ſame vertices with the cones, it may be demonſtrated that each pyramid in the firſt cone has to each in the other, taking them in the ſame order, the triplicate ratio of that which the ſide AK has to the ſide EM; that is, which AC has to EG. but as one antecedent to its conſequent, ſo are all the antecedents to all the conſequents [h]; h. 12. 5. therefore as the pyramid AKQL to the pyramid EMSN, ſo is the whole pyramid the baſe of which is the polygon DQATBYCV, and vertex L, to the whole pyramid of which the baſe is the polygon HSEOFPGR, and vertex N. wherefore alſo the firſt of theſe two laſt named pyramids has to the other the triplicate ratio of that which AC has to EG. but, by the Hypotheſis, the cone of which the baſe is the circle ABCD, and vertex L has to the ſolid X, the triplicate

Book XII. ratio of that which AC has to EG; therefore as the cone of which
the baſe is the circle ABCD, and vertex L, is to the ſolid X, ſo
is the pyramid the baſe of which is the polygon DQATBYCV,
and vertex L to the pyramid the baſe of which is the polygon
HSEOFPGR and vertex N. but the ſaid cone is greater than the
i. 14. 5. pyramid contained in it, therefore the ſolid X is greater [i] than the
pyramid the baſe of which is the polygon HSEOFPGR, and vertex
N. but it is alſo leſs; which is impoſſible. therefore the cone of
which the baſe is the circle ABCD, and vertex L has not to any





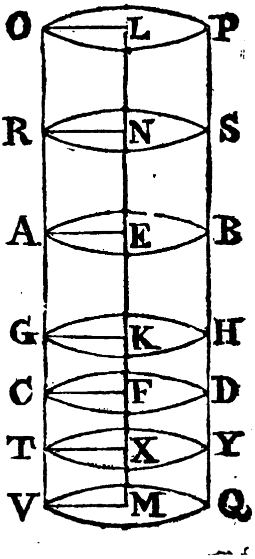
ſolid which is leſs than the cone of which the baſe is the circle EFGH and vertex N, the triplicate ratio of that which AC has to EG. In the ſame manner it may be demonſtrated that neither has the cone EFGHN to any ſolid which is leſs than the cone ABCDL, the triplicate ratio of that which EG has to AC. Nor can the cone ABCDL have to any ſolid which is greater than the cone EFGHN, the triplicate ratio of that which AC has to EG. for, if it be poſſible, let it have it to a greater, viz. to the ſolid Z. therefore, inverſely, the ſolid Z has to the cone ABCDL the triplicate ratio of that which EG has to AC. but as the ſolid Z is to the cone ABCDL, ſo is the cone EFGHN to ſome ſolid, which muſt be leſs [l] than the cone ABCDL, becauſe the ſolid Z is greater than the cone EFGHN. therefore the cone EFGHN has to a ſolid which is leſs than the cone ABCDL, the triplicate ratio of that which EG has to AC, which was demonſtrated to be impoſſible. therefore the cone ABCDL has not to any ſolid greater than the cone EFGHN, the triplicate ratio of that which AC has to EG; and it was demonſtrated that it could not have that ratio to any ſolid leſs than the cone EFGHN. therefore the cone ABCDL has to the cone EFGHN, the triplicate ratio of that which AC has to EG. but as the cone is to the cone, ſo [k] the cylinder to the cylinder, for every cone is the third part of the cylinder upon the ſame baſe, and of the ſame altitude. therefore alſo the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG. Wherefore ſimilar cones, &c. Q. E. D.

l. 14. 5.
k. 15. 5.

PROP. XIII. THEOR.

See N.

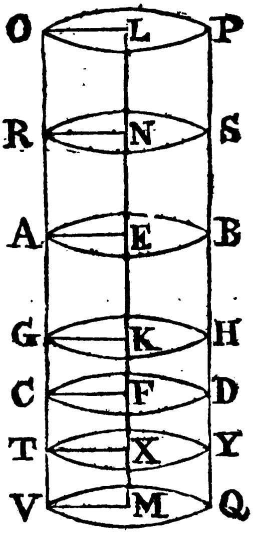
IF a cylinder be cut by a plane parallel to its oppoſite planes, or baſes; it divides the cylinder into two cylinders, one of which is to the other as the axis of the firſt to the axis of the other.



Let the cylinder AD be cut by the plane GH parallel to the oppoſite planes AB, CD, meeting the axis EF in the point K, and let the line GH be the common ſection of the plane GH and the ſurface of the cylinder AD. let AEFC be the parallelogram, in any poſition of it, by the revolution of which about the ſtraight line EF the cylinder AD is deſcribed; and let GK be the common ſection of the plane GH, and the plane AEFC. and becauſe the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common ſections with it, are parallel [a]; wherefore AK is a parallelogram, and GK equal to EA the ſtraight line from the center of the circle AB. for the ſame reaſon, each of the ſtraight lines

a. 16. 11.

drawn from the point K to the line GH may be proved to be equal to thoſe which are drawn from the center of the circle AB to its circumference, and are therefore all equal to one another. b. 15 Def. 1. therefore the line GH is the circumference of a circle [b] of which the center is the point K. therefore the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the ſame which would be deſcribed by the revolution of the parallelograms AK, GF about the ſtraight lines EK, KF. and it is to be ſhewn that the cylinder AH is to the cylinder HC, as the axis EK to the axis KF.



Produce the axis EF both ways; and take any number of ſtraight lines EN, NL, each equal to EK; and any number FX, XM, each equal to FK; and let planes parallel to AB, CD paſs through the points L, N, X, M. therefore the common ſections of theſe planes with the cylinder produced are circles the centers of which are the points L, N, X, M, as was proved of the plane GH; and theſe planes cut off the cylinders PR, RB, DT, TQ. and becauſe the axes LN, NE, EK are all equal, therefore c. 11. 12. the cylinders PR, RB, BG are [c] to one another as their baſes. but their baſes are equal, and therefore the cylinders PR, RB, BG are equal. and becauſe the axes LN, NE, EK are equal to one another, as alſo the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefore whatever multiple the axis KL is of the axis KE, the ſame multiple is the cylinder PG of the cylinder GB. for the ſame reaſon, whatever multiple the axis MK is of the axis KF, the ſame multiple is the cylinder QG of the cylinder GD. and if the axis KL be equal to the axis KM, the cylinder PG is equal to the cylinder GQ; and if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if leſs, leſs. ſince therefore there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD, and that of the axis EK and cylinder BG there has been taken any equimultiples whatever, viz. the axis KL and cylinder PG; and of the axis

KF and cylinder GD, any equimultiples whatever, viz. the axis KM and cylinder GQ; and it has been demonſtrated if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if equal, equal; and if leſs, leſs. therefore [d] the axis EK is to the axis KF, as the cylinder BG to the cylinder GD. Wherefore if a cylinder, &c. Q. E. D.

d. 5. Def. 5.

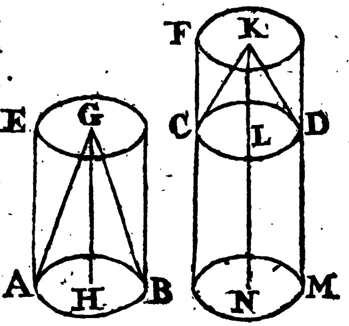
PROP. XIV. THEOR.

CONES and cylinders upon equal baſes are to one another as their altitudes.

Let the cylinders EB, FD be upon the equal baſes AB, CD. as the cylinder EB to the cylinder FD, ſo is the axis GH to the axis KL.

Produce the axis KL to the point N, and make LN equal to the axis GH, and let CM be a cylinder of which the baſe is CD, and axis LN. and becauſe the cylinders EB, CM have the ſame altitude, they are to one another as their baſes [a]. but their baſes are equal, therefore alſo the cylinders EB, CM are equal. and becauſe the cylinder FM is cut by the plane CD parallel to its oppoſite planes, as the cylinder CM to the cylinder FD, ſo is [b] the axis LN to the axis KL. but the cylinder CM is equal to the cylinder EB, and the axis LN to the axis GH. therefore as the cylinder EB to the cylinder FD, ſo is the axis GH to the axis KL. and as the cylinder EB to the cylinder FD, ſo is [c] the cone ABG to the cone CDK, becauſe the cylinders are triple [d] of the cones. therefore alſo the axis GH is to the axis KL, as the cone ABG to the cone CDK, and the cylinder EB to the cylinder FD. Wherefore cones, &c. Q. E. D.

a. 11. 12.
b. 13. 12.
c. 15. 5.
d. 10. 12.

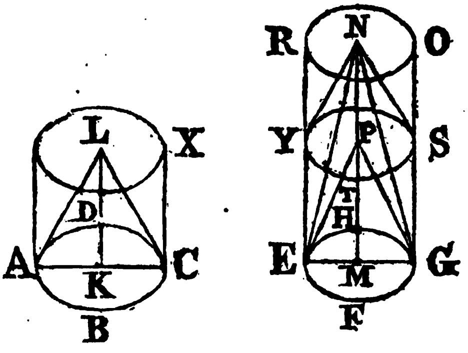


PROP. XV. THEOR.

See N. THE baſes and altitudes of equal cones and cylinders are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles ABCD, EFGH, the diameters of which are AC, EG, be the baſes, and KL, MN the axes, as alſo the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders. the baſes and altitudes of the cylinders AX, EO are reciprocally proportional; that is, as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL.

Either the altitude MN is equal to the altitude KL, or theſe altitudes are not equal. Firſt, let them be equal; and the cylinders AX, EO being alſo equal, and cones and cylinders of the ſame altitude
a. 11. 12. being to one another as their baſes[a]. therefore the baſe ABCD
b. A. 5. is equal[b] to the baſe EFGH; and as the baſe ABCD is to the baſe EFGH, ſo is the altitude MN to the altitude KL but let the altitudes KL, MN be unequal, and MN the greater of the two, and from MN take MP equal to KL, and, thro' the point P, cut the cylinder EO by the plane TYS parallel to the oppoſite planes of the circles EFGH, RO; therefore the common ſection of the plane TYS and the cylinder EO is a circle, and conſequently ES is a cylinder, the baſe of which is the circle EFGH, and altitude MP. and becauſe the cylinder AX is equal to the cylinder EO, as AX is to the cylinder ES, ſo[c] is the cylinder EO to
c. 7. 5.
the ſame ES. but as the cylinder AX to the cylinder ES, ſo[a] is the baſe ABCD to the baſe EFGH; for the cylinders AX, ES are of the ſame altitude; and as the cylinder EO to the cylinder ES,
d. 13. 12. ſo[d] is the altitude MN to the altitude MP, becauſe the cylinder



EO is cut by the plane TYS parallel to its oppoſite planes. therefore as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude MP. but MP is equal to the altitude KL; wherefore as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL; that is, the baſes and altitudes of the equal cylinders AX, EO are reciprocally proportional.

But let the baſes and altitudes of the cylinders AX, EO, be reciprocally proportional, viz. the baſe ABCD to the baſe EFGH, as the altitude MN to the altitude KL. the cylinder AX is equal to the cylinder EO.

Firſt, let the baſe ABCD be equal to the baſe EFGH, then becauſe as the baſe ABCD is to the baſe EFGH, ſo is the altitude MN to the altitude KL; MN is equal [b] to KL, and therefore the cylinder AX is equal [a] to the cylinder EO. b. A. 5. a. 11. 12.

But let the baſes ABCD, EFGH be unequal, and let ABCD be the greater; and becauſe as ABCD is to the baſe EFGH, ſo is the altitude MN to the altitude KL, therefore MN is greater [b] than KL; then, the ſame conſtruction being made as before, becauſe as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL; and becauſe the altitude KL is equal to the altitude MP; therefore the baſe ABCD is [a] to the baſe EFGH, as the cylinder AX to the cylinder ES; and as the altitude MN to the altitude MP or KL, ſo is the cylinder EO to the cylinder ES. therefore the cylinder AX is to the cylinder ES, as the cylinder EO is to the ſame ES. whence the cylinder AX is equal to the cylinder EO. and the ſame reaſoning holds in cones. Q. E. D.

PROP. XVI. PROB.

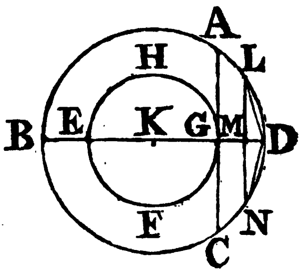
TO deſcribe in the greater of two circles that have the ſame center, a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle.

Let ABCD, EFGH be two given circles having the ſame center K. it is required to inſcribe in the greater circle ABCD a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle.

Thro' the center K draw the ſtraight line BD, and from the point G, where it meets the circumference of the leſſer circle, draw

GA at right angles to BD, and produce it to C; therefore AC touches [a] the circle EFGH. then if the circumference BAD be biſected, and the half of it be again biſected, and ſo on, there muſt at length remain a circumference leſs [b] than AD. let this be LD; and from the point L draw LM perpendicular to BD, and produce it to N; and join LD, DN. therefore LD is equal to DN, and becauſe LN is parallel to AC, and that AC touches the circle EFGH; therefore LN does not meet the circle EFGH. and much leſs ſhall the ſtraight lines LD, DN meet the circle EFGH. ſo that if ſtraight lines equal to LD be applied in the circle ABCD from the point L around to N, there ſhall be deſcribed in the circle a polygon of an even number of equal ſides not meeting the leſſer circle. Which was to be done.

a. 16. 3.
b. Lemma.



LEMMA II.

IF two trapeziums ABCD, EFGH be inſcribed in the circles the centers of which are the points K, L; and if the ſides AB, DC be parallel, as alſo EF, HG; and the other four ſides AD, BC, EH, FG be all equal to one another; but the ſide AB greater than EF, and DC greater than HG. the ſtraight line KA from the center of the circle in which the greater ſides are, is greater than the ſtraight line LE drawn from the center to the circumference of the other circle.

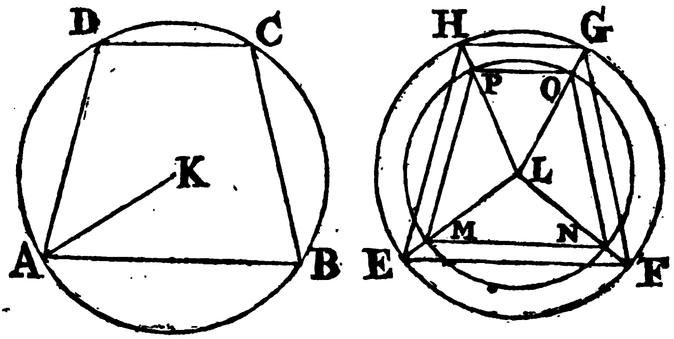
If it be poſſible, let KA be not greater than LE; then KA muſt be either equal to it, or leſs. Firſt, let KA be equal to LE. therefore becauſe in two equal circles, AD, BC in the one are equal to EH, FG in the other, the circumferences AD, BC are equal [a] to the circumferences EH, FG; but becauſe the ſtraight lines AB, DC are reſpectively greater than EF, GH, the circumferences AB, DC are greater than EF, HG. therefore the whole circumference ABCD is greater than the whole EFGH; but it is alſo equal to it,

a. 28. 3.

which is impoſſible. therefore the ſtraight line KA is not equal to LE.

But let KA be leſs than LE, and make LM equal to KA, and from the center L, and diſtance LM deſcribe the circle MNOP, meeting the ſtraight lines LE, LF, LG, LH, in M, N, O, P; and join MN, NO, OP, PM which are reſpectively parallel [b] to, and leſs than EF, FG, GH, HE. then, becauſe EH is greater than MP, AD is greater than MP; and the circles ABCD, MNOP are equal, therefore the circumference AD is greater than MP; for the ſame reaſon, the circumference BC is greater than NO; and becauſe the ſtraight line AB is greater than EF which is greater than MN, much more is AB greater than MN. therefore the circumference AB is greater than MN; and for the ſame reaſon, the circumference DC is greater than PO. therefore the whole circumference ABCD is greater than the whole MNOP; but it is likewiſe equal to it, which is impoſſible. therefore KA is not leſs than LE; nor is it equal to it; the ſtraight line KA muſt therefore be greater than LE. Q. E. D.

[b] 2. 6.



Cor. And if there be an Iſoſceles triangle the ſides of which are equal to AD, BC, but its baſe leſs than AB the greater of the two ſides AB, DC; the ſtraight line KA may, in the ſame manner, be demonſtrated to be greater than the ſtraight line drawn from the center to the circumference of the circle deſcribed about the triangle.

PROP. XVII. PROB.

See N.

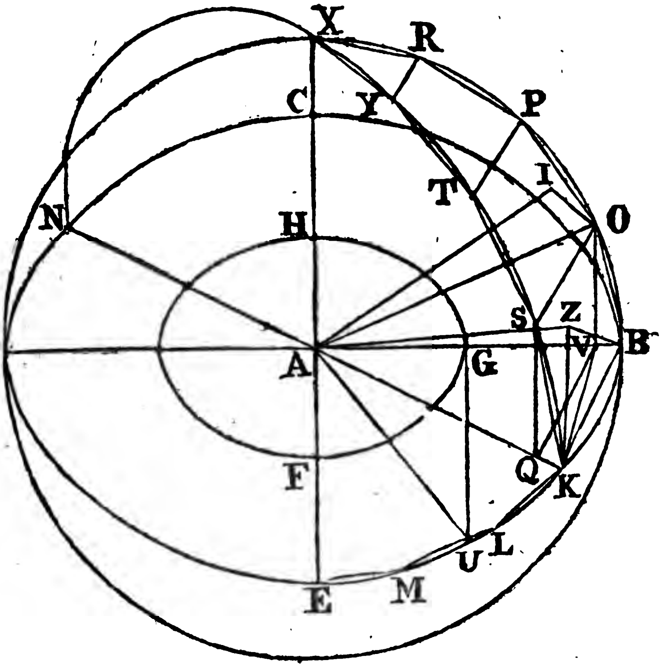
TO deſcribe in the greater of two ſpheres which have the ſame center, a ſolid polyhedron, the ſuperficies of which ſhall not meet the leſſer ſphere.

Let there be two ſpheres about the ſame center A; it is required to deſcribe in the greater a ſolid polyhedron the ſuperficies of which ſhall not meet the leſſer ſphere.

Let the ſpheres be cut by a plane paſſing thro' the center; the common ſections of it with the ſpheres ſhall be circles; becauſe the ſphere is deſcribed by the revolution of a ſemicircle about the diameter remaining unmoveable; ſo that in whatever poſition the ſemicircle be conceived, the common ſection of the plane in which it is with the ſuperficies of the ſphere is the circumference of a circle; and this is a great circle of the ſphere, becauſe the diameter of the ſphere which is likewiſe the diameter of the circle, is greater [a] than any ſtraight line in the circle or ſphere. let then the circle made by the ſection of the plane with the greater ſphere be BCDE, and with the leſſer ſphere be FGH; and draw the two diameters BD, CE at right angles to one another. and in BCDE the greater of the two circles deſcribe [b] a polygon of an even number of equal ſides not meeting the leſſer circle FGH; and let its ſides, in BE the fourth part of the circle, be BK, KL, LM, ME; join KA and produce it to N; and from A draw AX at right angles to the plane of the circle BCDE meeting the ſuperficies of the ſphere in the point X; and let planes paſs thro' AX and each of the ſtraight lines BD, KN, which, from what has been ſaid, ſhall produce great circles on the ſuperficies of the ſphere, and let BXD, KXN be the ſemicircles thus made upon the diameters BD, KN. therefore, becauſe XA is at right angles to the plane of the circle BCDE, every plane which paſſes thro' XA is at right [c] angles to the plane of the circle BCDE; wherefore the ſemicircles BXD, KXN are at right angles to that plane. and becauſe the ſemicircles BED, BXD, KXN, upon the equal diameters BD, KN are equal to one another, their fourth parts BE, BX, KX are equal to one another. therefore as many ſides of the polygon as are in the fourth part BE, ſo many there are in BX, KX equal to the ſides BK, KL, LM, ME. let theſe polygons be deſcribed, and their ſides be BO, OP, PR, RX; KS, ST, TY,

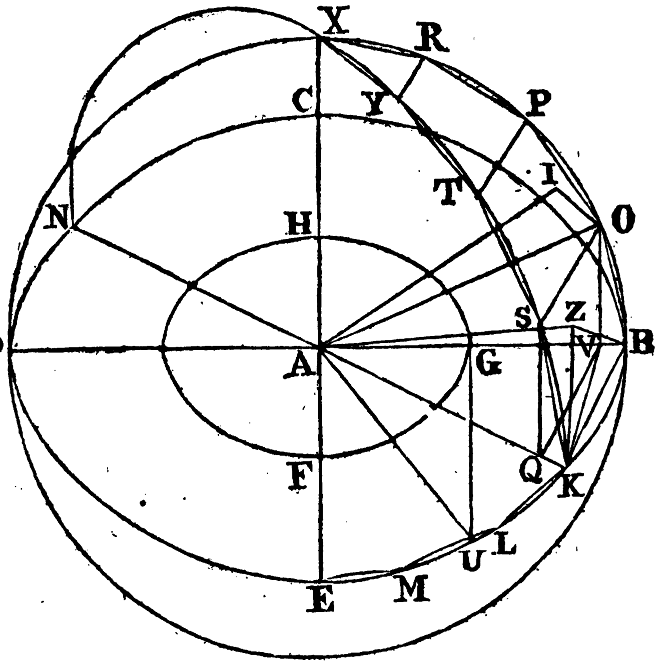
a. 15. 3.
b. 16. 12.
c. 18. 11.

YX, and join OS, PT, RY; and from the points O, S draw OV,
SQ perpendiculars to AB, AK. and becauſe the plane BOXD is at
right angles to the plane BCDE, and in one of them BOXD, OV
is drawn perpendicular to AB the common ſection of the planes,
therefore OV is perpendicular [d] to the plane BCDE. for the ſame d. 4. Def. 11.
reaſon SQ is perpendicular to the ſame plane, becauſe the plane
KSXN is at right angles to the plane BCDE. Join VQ, and becauſe in the equal ſemicircles BXD, KXN the circumferences BO,



KS are equal, and OV, SQ are perpendicular to their diameters,
therefore * OV is equal to SQ, and BV equal to KQ. but the * 26. 1.
whole BA is equal to the whole KA, therefore the remainder VA is
equal to the remainder QA. as therefore BV is to VA, ſo is KQ to
QA, wherefore VQ is parallel [e] to BK. and becauſe OV, SQ are e. 2. 6.
each of them at right angles to the plane of the circle BCDE, OV is
parallel [f] to SQ; and it has been proved that it is alſo equal to it; f. 6. 11.
therefore QV, SO are equal and parallel [g]. and becauſe QV is pa- g. 33. 1.
rallel to SO, and alſo to KB; OS is parallel [h] to BK; and therefore h. 9. 11.

Book XII. BO, KS which join them are in the ſame plane in which theſe parallels are, and the quadrilateral figure KBOS is in one plane. and if PB, TK be joined, and perpendiculars be drawn from the points P, T to the ſtraight lines AB, AK, it may be demonſtrated that TP is parallel to KB in the very ſame way that SO was ſhewn to be parallel h. 9. 11. to the ſame KB; wherefore [h] TP is parallel to SO, and the quadrilateral figure SOPT is in one plane. for the ſame reaſon the quadrilateral TPRY is in one plane. and the figure YRX is alſo in one i. 2. 11. plane [i]. therefore, if from the points O, S, P, T, R, Y there be drawn ſtraight lines to the point A, there ſhall be formed a ſolid polyhedron between the circumferences BX, KX compoſed of pyramids the baſes of which are the quadrilaterals KBOS, SOPT, TPRY, and the triangle YRX, and of which the common vertex is the point A. and if the ſame conſtruction be made upon each of the ſides KL, LM, ME, as has been done upon BK, and the like be done alſo in the other three quadrants, and in the other hemiſphere; there ſhall be formed a ſolid polyhedron deſcribed in the ſphere,



compoſed of pyramids the baſes of which are the aforeſaid quadrilateral figures, and the triangle YRX, and thoſe formed in the like manner in the reſt of the ſphere, the common vertex of them all being the point A. and the ſuperficies of this ſolid polyhedron does not meet the leſſer ſphere in which is the circle FGH. for from the point A draw [k] AZ perpendicular to the plane of the quadrilateral KBOS meeting it in Z, and join BZ, ZK. and becauſe AZ is perpendicular to the plane KBOS, it makes right angles with every ſtraight line meeting it in that plane; therefore AZ is perpendicular to BZ and ZK. and becauſe AB is equal to AK, and that the ſquares of AZ, ZB, are equal to the ſquare of AB; and the ſquares of AZ, ZK to the ſquare of AK *; therefore the ſquares of AZ, ZB are equal to the ſquares of AZ, ZK. take from theſe equals the ſquare of AZ, the remaining ſquare of BZ is equal to the remaining ſquare of ZK; and therefore the ſtraight line BZ is equal to ZK. in the like manner it may be demonſtrated that the ſtraight lines drawn from the point Z to the points O, S are equal to BZ, or ZK. therefore the circle deſcribed from the center Z, and diſtance ZB ſhall paſs thro' the points K, O, S, and KBOS ſhall be a quadrilateral figure in the circle. and becauſe KB is greater than QV, and QV equal to SQ, therefore KB is greater than SO. but KB is equal to each of the ſtraight lines BO, KS; wherefore each of the circumferences cut off by KB, BO, KS is greater than that cut off by OS; and theſe three circumferences together with a fourth equal to one of them, are greater than the ſame three together with that cut off by OS; that is, than the whole circumference of the circle; therefore the circumference ſubtended by KB is greater than the fourth part of the whole circumference of the circle KBOS, and conſequently the angle BZK at the center is greater than a right angle. and becauſe the angle BZK is obtuſe, the ſquare of BK is greater [l] than the ſquares of BZ, ZK; that is, greater than twice the ſquare of BZ. Join KV, and becauſe in the triangles KBV, OBV, KB, BV are equal to OB, BV, and that they contain equal angles; the angle KVB is equal [m] to the angle OVB. and OVB is a right angle; therefore alſo KVB is a right angle. and becauſe BD is leſs than twice DV, the rectangle contained by DB, BV is leſs than twice the rectangle DVB; that is [n], the ſquare of KB is leſs than twice the ſquare of KV. but the ſquare of KB is greater than twice the ſquare of BZ; therefore the ſquare of KV is greater than the ſquare of BZ. and becauſe BA is equal to AK, and that the ſquares of

k. 11. 11.

* 47. 1.

l. 12. 2.

m. 4. 1.

n. 8. 6.

BZ, ZA are equal together to the ſquare of BA, and the ſquares of KV, VA to the ſquare of AK; therefore the ſquares of BZ, ZA are equal to the ſquares of KV, VA; and of theſe the ſquare of KV is greater than the ſquare of BZ, therefore the ſquare of VA is leſs than the ſquare of ZA, and the ſtraight line AZ greater than VA. much more then AZ is greater than AG, becauſe in the preceding Propoſition it was ſhewn that KV falls without the circle FGH. and AZ is perpendicular to the plane KBOS, and is therefore the ſhorteſt of all the ſtraight lines that can be drawn from A the center of the ſphere to that plane. Therefore the plane KBOS does not meet the leſſer ſphere.

And that the other planes between the quadrants BX, KX fall without the leſſer ſphere, is thus demonſtrated. from the point A draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and as was demonſtrated of the plane KBOS and the point Z, in the ſame way it may be ſhewn that the point I is the center of a circle deſcribed about SOPT, and that OS is greater than PT; and PT was ſhewn to be parallel to OS. therefore becauſe the two trapeziums KBOS, SOPT inſcribed in circles have their ſides BK, OS parallel, as alſo OS, PT; and their other ſides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS greater than PT, therefore the ſtraight line ZB is greater ° than IO. Join AO which will be equal to AB; and becauſe AIO, AZB are right angles, the ſquares of AI, IO are equal to the ſquare of AO or of AB; that is, to the ſquares of AZ, ZB; and the ſquare of ZB is greater than the ſquare of IO, therefore the ſquare of AZ is leſs than the ſquare of AI; and the ſtraight line AZ leſs than the ſtraight line AI. and it was proved that AZ is greater than AG; much more then is AI greater than AG. therefore the plane SOPT falls wholly without the leſſer ſphere. in the ſame manner it may be demonſtrated that the plane TPRY falls without the ſame ſphere, as alſo the triangle YRX, viz. by the Cor. of 2d Lemma. and after the ſame way it may be demonſtrated that all the planes which contain the ſolid polyhedron fall without the leſſer ſphere. therefore in the greater of two ſpheres which have the ſame center, a ſolid polyhedron is deſcribed the ſuperficies of which does not meet the leſſer ſphere. Which was to be done.

o. 2. Lem. 12.

But the ſtraight line AZ may be demonſtrated to be greater than AG otherwiſe and in a ſhorter manner, without the help of Prop. 16. as follows. From the point G draw GU at right angles to AG and join AU. if then the circumference BE be biſected, and its half again biſected, and ſo on, there will at length be left a circumference leſs than the circumference which is ſubtended by a ſtraight line equal to GU inſcribed in the circle BCDE. let this be the circumference KB. therefore the ſtraight line KB is leſs than GU. and becauſe the angle BZK is obtuſe, as was proved in the preceding, therefore BK is greater than BZ. but GU is greater than BK; much more then is GU greater than BZ, and the ſquare of GU than the ſquare of BZ. and AU is equal to AB; therefore the ſquare of AU, that is the ſquares of AG, GU are equal to the ſquare of AB, that is to the ſquares of AZ, ZB; but the ſquare of BZ is leſs than the ſquare of GU; therefore the ſquare of AZ is greater than the ſquare of AG, and the ſtraight line AZ conſequently greater than the ſtraight line AG.

Cor. And if in the leſſer ſphere there be deſcribed a ſolid polyhedron by drawing ſtraight lines betwixt the points in which the ſtraight lines from the center of the ſphere drawn to all the angles of the ſolid polyhedron in the greater ſphere meet the ſuperficies of the leſſer; in the ſame order in which are joined the points in which the ſame lines from the center meet the ſuperficies of the greater ſphere; the ſolid polyhedron in the ſphere BCDE has to this other ſolid polyhedron the triplicate ratio of that which the diameter of the ſphere BCDE has to the diameter of the other ſphere. for if theſe two ſolids be divided into the ſame number of pyramids, and in the ſame order; the pyramids ſhall be ſimilar to one another, each to each. becauſe they have the ſolid angles at their common vertex, the center of the ſphere, the ſame in each pyramid, and their other ſolid angles at the baſes equal to one another, each to each [a], becauſe they are contained by three plane angles equal each to each; and the pyramids are contained by the ſame number of ſimilar planes; and are therefore ſimilar [b] to one another, each to each. but ſimilar pyramids have to one another the triplicate [c] ratio of their homologous ſides. therefore the pyramid of which the baſe is the quadrilateral KBOS, and vertex A, has to the pyramid in the other ſphere of the ſame order, the triplicate ratio of their homologous ſides; that is, of that ratio which AB from the center of the greater ſphere has to the ſtraight line from the ſame center to the ſuperficies of the leſſer ſphere. and in like manner each pyramid in the greater ſphere has to each of the ſame order in the leſſer, the triplicate ratio of that which AB has to the ſemidiameter of the leſſer ſphere. and as one

[a] B. 11.

[b] 11. Def. 11.

[c] Cor. 8. 12.

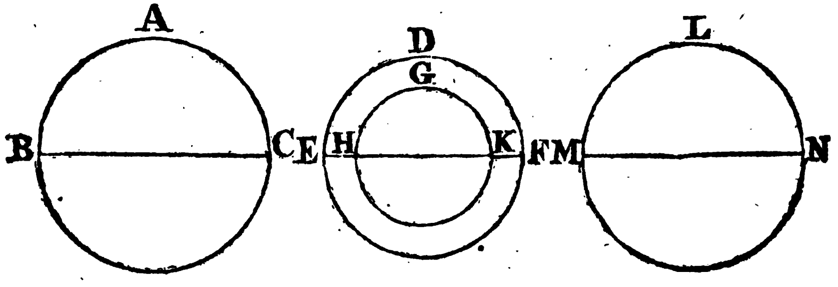
Book XII. antecedent is to its conſequent, ſo are all the antecedents to all the conſequents. Wherefore the whole ſolid polyhedron in the greater ſphere has to the whole ſolid polyhedron in the other, the triplicate ratio of that which AB the ſemidiameter of the firſt has to the ſemidiameter of the other; that is, which the diameter BD of the greater has to the diameter of the other ſphere.

PROP. XVIII. THEOR.

SPHERES have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two ſpheres of which the diameters are BC, EF. the ſphere ABC has to the ſphere DEF the triplicate ratio of that which BC has to EF.

For if it has not, the ſphere ABC ſhall have to a ſphere either leſs or greater than DEF, the triplicate ratio of that which BC has to EF. Firſt, let it have that ratio to a leſs, viz. to the ſphere GHK; and let the ſphere DEF have the ſame center with GHK; a. 17. 12. and in the greater ſphere DEF deſcribe [a] a ſolid polyhedron the ſuperficies of which does not meet the leſſer ſphere GHK; and in the ſphere ABC deſcribe another ſimilar to that in the ſphere DEF. therefore the ſolid polyhedron in the ſphere ABC has to b. Cor. 17. 12. the ſolid polyhedron in the ſphere DEF, the triplicate ratio [b] of that which BC has to EF. but the ſphere ABC has to the ſphere GHK, the triplicate ratio of that which BC has to EF; therefore as the ſphere ABC to the ſphere GHK, ſo is the ſolid polyhedron in the ſphere ABC to the ſolid polyhedron in the ſphere DEF. but the ſphere ABC is greater than the ſolid polyhedron in it;



therefore [c] alſo the ſphere GHK is greater than the ſolid polyhedron in the ſphere DEF. but it is alſo leſs, becauſe it is contained within it, which is impoſſible. therefore the ſphere ABC has not to any ſphere leſs than DEF, the triplicate ratio of that which BC has to EF. In the ſame manner it may be demonſtrated that the ſphere DEF has not to any ſphere leſs than ABC, the triplicate ratio of that which EF has to BC. Nor can the ſphere ABC have to any ſphere greater than DEF, the triplicate ratio of that which BC has to EF. for if it can, let it have that ratio to a greater ſphere LMN. therefore, by inverſion, the ſphere LMN has to the ſphere ABC, the triplicate ratio of that which the diameter EF has to the diameter BC. but as the ſphere LMN to ABC, ſo is the ſphere DEF to ſome ſphere, which muſt be leſs [c] than the ſphere ABC, becauſe the ſphere LMN is greater than the ſphere DEF. therefore the ſphere DEF has to a ſphere leſs than ABC the triplicate ratio of that which EF has to BC; which was ſhewn to be impoſſible. therefore the ſphere ABC has not to any ſphere greater than DEF the triplicate ratio of that which BC has to EF. and it was demonſtrated that neither has it that ratio to any ſphere leſs than DEF. Therefore the ſphere ABC has to the ſphere DEF, the triplicate ratio of that which BC has to EF. Q. E. D.

c. 14. 5.

FINIS.

NOTES

CRITICAL AND GEOMETRICAL;

CONTAINING,

AN ACCOUNT OF THOSE THINGS IN WHICH THIS EDITION
DIFFERS FROM THE GREEK TEXT; AND THE REASONS
OF THE ALTERATIONS WHICH HAVE BEEN MADE.

AS ALSO

OBSERVATIONS ON SOME OF THE PROPOSITIONS.

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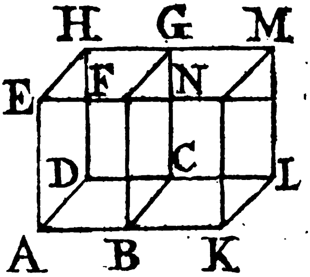
NOTES, &c.

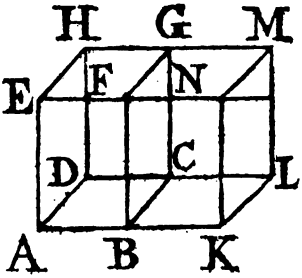
DEFINITION I. BOOK I.

IT is neceſſary to conſider a ſolid, that is a magnitude which has length, breadth and thickneſs, in order to underſtand aright the Definitions of a point, line and ſuperficies; for theſe all ariſe from a ſolid, and exiſt in it. the boundary, or boundaries which contain a ſolid are called ſuperficies, or the boundary which is common to two ſolids which are contiguous, or which divides one ſolid into two contiguous parts, is called a ſuperficies. thus if BCGF be one of the boundaries which contain the ſolid ABCDEFGH, or which is the common boundary of this ſolid, and the ſolid BKLCFNMG and is therefore in the one as well as the other ſolid, it is called a ſuperficies, and has no thickneſs. for if it have any, this thickneſs muſt either be a part of the thickneſs of the ſolid AG, or of the ſolid BM, or a part of the thickneſs of each of them. It cannot be a part of the thickneſs of the ſolid BM, becauſe if this ſolid be removed from the ſolid AG, the ſuperficies BCGF, the boundary of the ſolid AG, remains ſtill the ſame as it was. Nor can it be a part of the thickneſs of the ſolid AG, becauſe if this be removed from the ſolid BM, the ſuperficies BCGF, the boundary of the ſolid BM, does nevertheleſs remain. therefore the ſuperficies BCGF has no thickneſs; but only length and breadth.



The boundary of a ſuperficies is called a line, or a line is the common boundary of two ſuperficies that are contiguous, or which divides one ſuperficies into two contiguous parts. thus if BC be one of the boundaries which contain the ſuperficies ABCD, or which is the common boundary of this ſuperficies and of the ſuperficies KBCL which is contiguous to it, this boundary BC is called a line, and has no breadth. for if it have any, this muſt be part either of the breadth of the ſuperficies ABCD, or of the ſuperficies KBCL, or part of each of them. It is not part of the breadth of the ſuperficies KBCL, for if this ſuperficies be removed from the ſuperficies ABCD, the line BC which is the boundary of the ſuperficies ABCD remains the ſame as it was. nor can the breadth that BC is ſuppoſed to have be a part of the breadth of the ſuperficies ABCD, becauſe if this be removed from the ſuperficies KBCL, the line BC which is the boundary of the ſuperficies KBCL does nevertheleſs remain. therefore the line BC has no breadth. and becauſe the line BC is in a ſuperficies, and that a ſuperficies has no thickneſs, as was ſhewn; therefore a line has neither breadth nor thickneſs, but only length.

The boundary of a line is called a point, or a point is the common boundary or extremity of two lines that are contiguous. thus if B be the extremity of the line AB, or the common extremity of the two lines AB, KB, this extremity is called a point, and has no length. for if it have any, this length muſt either be part of the length of the line AB, or of the line KB. It is not part of the length of KB, for if the line KB be removed from AB, the point B which is the extremity of the line AB remains the ſame as it was. nor is it part of the length of the line AB; for if AB be removed from the line KB, the point B which is the extremity of the line KB, does nevertheleſs remain. therefore the point B has no length. and becauſe a point is in a line, and a line has neither breadth nor thickneſs, therefore a point has no length, breadth nor thickneſs. And in this manner the Definitions of a point, line and ſuperficies are to be underſtood.



DEF. VII. B. I.

Inſtead of this Definition as it is in the Greek copies, a more diſtinct one is given from a property of a plane ſuperficies, which is manifeſtly ſuppoſed in the Elements, viz. that a ſtraight line drawn from any point in a plane to any other in it, is wholly in that plane.

DEF. VIII. B. I.

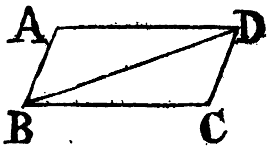
It ſeems that he who made this Definition deſigned that it ſhould comprehend not only a plane angle contained by two ſtraight lines, but likewiſe the angle which ſome conceive to be made by a ſtraight line and a curve, or by two curve lines, which meet one another in a plane. but tho' the meaning of the words ἐπ' εὐθείας; that is, in a ſtraight line, or in the ſame direction, be plain, when two ſtraight lines are ſaid to be in a ſtraight line, it does not appear what ought to be underſtood by theſe words, when a ſtraight line and a curve, or two curve lines, are ſaid to be in the ſame direction; at leaſt it cannot be explained in this place; which makes it probable that this Definition, and that of the angle of a ſegment, and what is ſaid of the angle of a ſemicircle, and the angles of ſegments, in the 16. and 31. Propoſitions of Book 3. are the additions of ſome leſs ſkilful Editor. on which account, eſpecially ſince they are quite uſeleſs, theſe Definitions are diſtinguiſhed from the reſt by inverted double commas.

DEF. XVII. B. I.

The words "which alſo divides the circle into two equal parts" are added at the end of this Definition in all the copies, but are now left out as not belonging to the Definition, being only a Corollary from it. Proclus demonſtrates it by conceiving one of the parts into which the diameter divides the circle, to be applied to the other, for it is plain they muſt coincide, elſe the ſtraight lines from the center to the circumference would not be all equal. the ſame thing is eaſily deduced from the 31. Prop. of Book 3. and the 24. of the ſame; from the firſt of which it follows that ſemicircles are ſimilar ſegments of a circle. and from the other, that they are equal to one another.

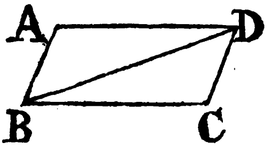
DEF. XXXIII. B. I.

This Definition has one condition more than is neceſſary; becauſe every quadrilateral figure which has its oppoſite ſides equal to one another, has likewiſe its oppoſite angles equal; and on the contrary.



Let ABCD be a quadrilateral figure, of which the oppoſite ſides AB, CD are equal to one another; as alſo AD and BC. join BD; the two ſides AD, DB are equal to the two CB, BD, and the baſe AB is equal to the baſe CD; therefore by Prop. 8. of Book 1. the angle ADB is equal to the angle CBD; and by Prop. 4. B. 1. the angle BAD is equal to the angle DCB, and ABD to BDC; and therefore alſo the angle ADC is equal to the angle ABC.

And if the angle BAD be equal to the oppoſite angle BCD, and the angle ABC to ADC; the oppoſite ſides are equal. Becauſe by Prop. 32. B. 1. all the angles of the quadrilateral figure ABCD are together equal to four right angles, and the two angles BAD, ADC are together equal to the two angles BCD, ABC. wherefore BAD, ADC are the half of all the four angles; that is, BAD and ADC are equal to two right angles. and therefore AB, CD are parallels by Prop. 28. B. 1. in the ſame manner AD, BC are parallels. therefore ABCD is a parallelogram, and its oppoſite ſides are equal by 34. Prop. B. 1.



PROP. VII. B. I.

There are two caſes of this Propoſition, one of which is not in the Greek text, but is as neceſſary as the other. and that the caſe left out has been formerly in the text appears plainly from this, that the ſecond part of Prop 5. which is neceſſary to the Demonſtration of this caſe, can be of no uſe at all in the Elements, or any where elſe, but in this Demonſtration; becauſe the ſecond part of Prop. 5. clearly follows from the firſt part, and Prop. 13. B. 1. this part muſt therefore have been added to Prop. 5. upon account of ſome Propoſition betwixt the 5. and 13. but none of theſe ſtand in need of it, except the 7. Propoſition, on account of which it has been added. beſides the tranſlation from the Arabic has this caſe explicitely demonſtrated. and Proclus acknowledges that the ſecond part of Prop. 5. was added upon account of Prop. 7. but gives a ridiculous reaſon for it, " that it might afford an anſwer " to objections made againſt the 7." as if the caſe of the 7. which is left out, were, as he expreſsly makes it, an objection againſt the propoſition itſelf. Whoever is curious may read what Proclus ſays of this in his commentary on the 5. and 7. Propoſitions; for it is not worth while to relate his trifles at full length.

It was thought proper to change the enuntiation of this 7. Prop. ſo as to preſerve the very ſame meaning; the literal tranſlation from the Greek being extremely harſh, and difficult to be underſtood by beginners.

PROP. XI. B. I.

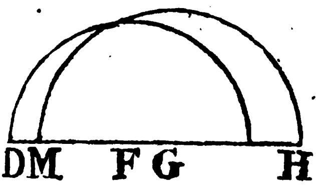
A Corollary is added to this Propoſition, which is neceſſary to Prop. 1. B. 11. and otherwiſe.

PROP. XX. and XXI. B. I.

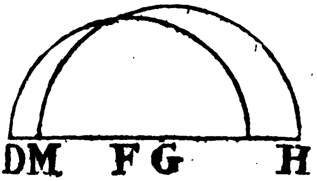
Proclus in his Commentary relates that the Epicureans derided this Propoſition, as being manifeſt even to aſſes, and needing no Demonſtration; and his anſwer is, that tho' the truth of it be manifeſt to our ſenſes, yet it is ſcience which muſt give the reaſon why two ſides of a triangle are greater than the third. but the right anſwer to this objection againſt this and the 21. and ſome other plain Propoſitions, is, that the number of Axioms ought not to be encreaſed without neceſſity, as it muſt be if theſe Propoſitions be not demonſtrated. Monſ. Clairault in the Preface to his Elements of Geometry publiſhed in French at Paris Ann. 1741. ſays that Euclid has been at the pains to prove that the two ſides of a triangle which is included within another are together leſs than the two ſides of the triangle which includes it; but he has forgot to add this condition, viz. that the triangles muſt be upon the ſame baſe; becauſe unleſs this be added, the ſides of the included triangle may be greater than the ſides of the triangle which includes it, in any ratio which is leſs than that of two to one. as Pappus Alexandrinus has demonſtrated in Prop. 3. B. 3. of his Mathematical Collections.

PROP. XXII. B. I.

Some Authors blame Euclid becauſe he does not demonſtrate that the two circles made uſe of in the conſtruction of this Problem muſt cut one another. but this is very plain from the determination he has given, viz. that any two of the ſtraight lines DF, FG, GH muſt be greater than the third. for who is ſo dull, tho' only beginning to learn the Elements, as not to perceive that the circle deſcribed from the center F, at the diſtance FD, muſt meet FH betwixt F and H, becauſe FD is leſs than FH; and that, for the like reaſon, the circle deſcribed from the center G, at the diſtance GH or GM muſt meet DG betwixt D and G; and

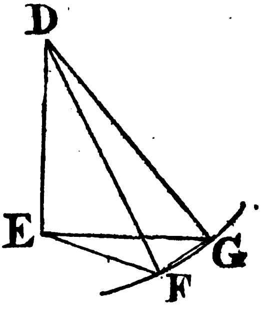


that theſe circles muſt meet one another, becauſe FD and GH are together greater than FG? and this determination is eaſier to be underſtood than that which Mr. Thomas Simpſon derives from it, and puts inſtead of Euclid's, in the 49. page of his Elements of Geometry, that he may ſupply the omiſſion he blames Euclid for; which determination is, that any of the three ſtraight lines muſt be leſs than the ſum, but greater than the difference of the other two. from this he ſhews the circles muſt meet one another, in one caſe; and ſays that it may be proved after the ſame manner in any other caſe. but the ſtraight line GM which he bids take from GF may be greater than it, as in the figure here annexed, in which caſe his demonſtration muſt be changed into another.



PROP. XXIV. B. I.

To this is added "of the two ſides DE, DF, let DE be that "which is not greater than the other;" that is, take that ſide of the two DE, DF which is not greater than the other, in order to make with it the angle EDG equal to BAC. becauſe without this reſtriction, there might be three different caſes of the Propoſition, as Campanus and others make.



Mr. Thomas Simpſon in p. 262. of the ſecond edition of his Elements of Geometry printed Ann. 1760. obſerves in his Notes, that it ought to have been ſhewn that the point F falls below the line EG; this probably Euclid omitted, as it is very eaſy to perceive that DG being equal to DF, the point G is in the circumference of a circle deſcribed from the center D at the diſtance DF, and muſt be in that part of it which is above the ſtraight line EF, becauſe DG falls above DF, the angle EDG being greater than the angle EDF.

PROP. XXIX. B. I.

The Propoſition which is uſually called the 5. Poſtulate, or 11. Axiom, by ſome the 12. on which this 29. depends, has given a great deal to do both to antient and modern Geometers. it ſeems not to be properly placed among the Axioms, as, indeed, it is not ſelf-evident; but it may be demonſtrated thus.

DEFINITION 1.

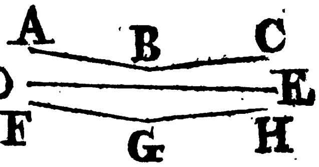
The diſtance of a point from a ſtraight line, is the perpendicular drawn to it from the point.

DEF. 2.

One ſtraight line is ſaid to go nearer to, or further from another ſtraight line, when the diſtance of the points of the firſt from the other ſtraight line become leſs or greater than they were; and two ſtraight lines are ſaid to keep the ſame diſtance from one another, when the diſtance of the points of one of them from the other is always the ſame.

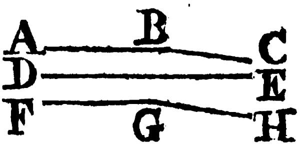
AXIOM.

A ſtraight line cannot firſt come nearer to another ſtraight line, and then go further from it, before it cuts it; and, in like manner, a ſtraight line cannot go further from another ſtraight line, and then come nearer to it; nor can a ſtraight line keep the ſame diſtance from another ſtraight line, and then come nearer to it, or go further from it; or a ſtraight line keep always the ſame direction.



For example, the ſtraight line ABC cannot firſt come nearer to the ſtraight line DE, as from the point A to the point B, and then, from the point B to the point C, go further from the ſame DE. and, in like manner, the ſtraight line FGH cannot go further from DE, as from F to G, and then, from G to H, come nearer to the ſame DE. and ſo in the laſt caſe as in fig. 2.

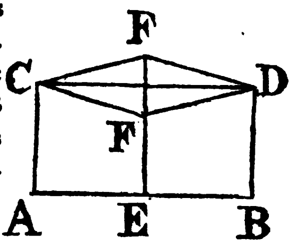
See the figure above.



PROP. 1.

If two equal ſtraight lines AC, BD be each at right angles to the ſame ſtraight line AB; if the points C, D be joined by the ſtraight line CD, the ſtraight line EF drawn from any point E in AB unto CD, at right angles to AB, ſhall be equal to AC, or BD.

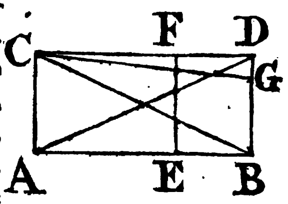
If EF be not equal to AC, one of them muſt be greater than the other; let AC be the greater; then becauſe FE is leſs than CA, the ſtraight line CFD is nearer to the ſtraight line AB at the point F than at the point C, that is CF comes nearer to AB from the point C to F. but becauſe DB is greater than FE, the ſtraight line CFD is further from AB at the point D than at F, that is FD goes further from AB from F to D. therefore the ſtraight line CED firſt comes nearer to the ſtraight line AB, and then goes further from it, before it cuts it, which is impoſſible. and the ſame thing will follow, if FE be ſaid to be greater than CA, or DB. therefore FE is not unequal to AC, that is, it is equal to it.



PROP. 2.

If two equal ſtraight lines AC, BD be each at right angles to the ſame ſtraight line AB; the ſtraight line CD which joins their extremities makes right angles with AC and BD.

Join AD, BC; and becauſe in the triangles CAB, DBA, CA, AB are equal to DB, BA, and the angle CAB equal to the angle
a. 4. 1. DBA; the baſe BC is equal[a] to the baſe AD. and in the triangles ACD, BDC, AC, CD are equal to BD, DC, and the baſe AD is equal to the baſe BC, therefore the an-
b. 8. 1. gle ACD is equal[b] to the angle BDC. from any point E in AB draw EF unto CD, at right angles to AB; therefore, by Prop. 1. EF is equal to AC, or BD; wherefore, as has been juſt now ſhewn, the angle ACF is equal to the angle EFC. in the ſame manner the angle BDF is equal to the angle EFD; but the angles ACD, BDC are equal, therefore the angles
c. 10. Def. 1. EFC and EFD are equal, and right angles[c]; wherefore alſo the angles ACD, BDC are right angles.



Cor. Hence, if two ſtraight lines AB, CD be at right angles to the ſame ſtraight line AC, and if betwixt them a ſtraight line BD be drawn at right angles to either of them, as to AB; then BD is equal to AC, and BDC is a right angle.

If AC be not equal to BD, take BG equal to AC, and join CG. therefore, by this Propoſition, the angle ACG is a right angle; but ACD is alſo a right angle, wherefore the angles ACD, ACG are equal to one another, which is impoſſible. therefore BD is equal to AC; and by this Propoſition BDC is a right angle.

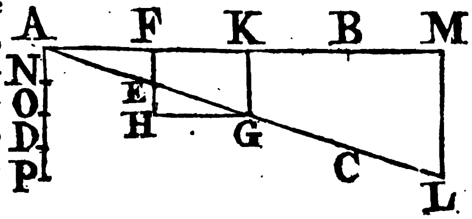
PROP. 3.

If two ſtraight lines which contain an angle be produced, there may be found in either of them a point from which the perpendicular drawn to the other ſhall be greater than any given ſtraight line.

Let AB, AC be two ſtraight lines which make an angle with one another, and let AD be the given ſtraight line; a point may be found either in AB or AC, as in AC, from which the perpendicular drawn to the other AB ſhall be greater than AD.

In AC take any point E, and draw EF perpendicular to AB; produce AE to G ſo that EG be equal to AE; and produce FE to H, and make EH equal to FE, and join HG. becauſe, in the triangles AEF, GEH, AE, EF are equal to GE, EH, each to each, and contain equal [a] angles, the angle GHE is therefore equal [b] to the angle AFE which is a right angle. draw GK perpendicular to AB; and becauſe the ſtraight lines FK, HG are at right angles to FH, and KG at right angles to FK; KG is equal to FH, by Cor. Pr. 2. that is to the double of FE. in the ſame manner, if AG be produced to L ſo that GL be equal to AG, and LM be drawn perpendicular to AB, then LM is double of GK, and ſo on. In AD take AN equal to FE, and AO equal to KG, that is to the double of FE, or AN; alſo take AP equal to LM, that is to the double of KG, or AO; and let this be done till the ſtraight line taken be greater than AD. let this ſtraight line ſo taken be AP, and becauſe AP is equal to LM, therefore LM is greater than AD. Which was to be done.

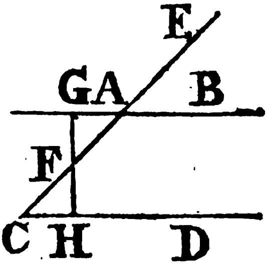
a. 15. 1.
b. 4. 1.



PROP. 4.

If two ſtraight lines AB, CD make equal angles EAB, ECD with another ſtraight line EAC towards the ſame parts of it; AB and CD are at right angles to ſome ſtraight line.

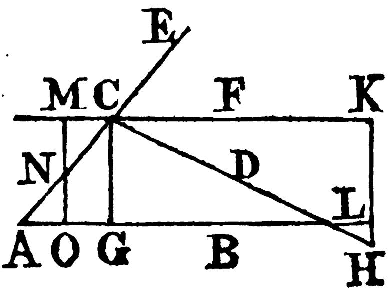
Biſect AC in F, and draw FG perpendicular to AB; take CH in the ſtraight line CD equal to AG and on the contrary ſide of AC to that on which AG is, and join FH. therefore, in the triangles AFG, CFH the ſides FA, AG are equal to FC, CH, each to each, and a. 15. 1. the angle FAG, that [a] is EAB is equal to b. 4. 1. the angle FCH; wherefore [b] the angle AGF is equal to CHF, and AFG to the angle CFH. to theſe laſt add the common angle AFH, therefore the two angles AFG, AFH are equal to the two angles CFH, HFA which two laſt are equal to- c. 13. 1. gether to two right angles [c], therefore alſo AFG, AFH are equal to two right d. 14. 1. angles, and conſequently [d] GF and FH are in one ſtraight line. and becauſe AGF is a right angle, CHF which is equal to it is alſo a right angle. therefore the ſtraight lines AB, CD are at right angles to GH



PROP. 5.

If two ſtraight lines AB, CD be cut by a third ACE ſo as to make the interior angles BAC, ACD, on the ſame ſide of it, together leſs than two right angles; AB and CD being produced ſhall meet one another towards the parts on which are the two angles which are leſs than two right angles.

a. 23. 1. At the point C in the ſtraight line CE make [a] the angle ECF equal to the angle EAB, and draw to AB the ſtraight line CG at right angles to CF. then becauſe the angles ECF, EAB are equal to one another, and that the angles ECF, FCA are toge- b. 13. 1. ther equal [b] to two right angles, the angles EAB, FCA are equal to two right angles. but, by the hypotheſis, the angles EAB, ACD are together leſs than two right angles, therefore the angle FCA is greater than ACD, and CD falls between CF and AB. and becauſe CF and CD make an angle with one another, by Prop. 3. a point may be found in either of them CD from which the perpendicular drawn to CF ſhall be greater than the ſtraight line CG. let this point be H, and draw HK perpendicular to CF meeting AB in L. and becauſe AB, CF contain equal angles with AC on the ſame ſide of it, by Prop. 4. AB and



CF are at right angles to the ſtraight line MNO which biſects AC in N and is perpendicular to CF. therefore, by Cor. Prop. 2. CG and KL which are at right angles to CF are equal to one another. and HK is greater than CG, and therefore is greater than KL, and conſequently the point H is in KL produced. Wherefore the ſtraight line CDH drawn betwixt the points C, H which are on contrary ſides of AL, muſt neceſſarily cut the ſtraight line AB.

PROP. XXXV. B. I.

The Demonſtration of this Propoſition is changed, becauſe if the method which is uſed in it was followed, there would be three caſes to be ſeparately demonſtrated, as is done in the tranſlation from the Arabic; for in the Elements no caſe of a Propoſition that requires a different demonſtration ought to be omitted. On this account we have choſen the method which Monſ. Clairault has given, the firſt of any, as far as I know, in his Elements, page 21. and which afterwards Mr. Simpſon gives in his, page 32. but whereas Mr. Simpſon makes uſe of Prop. 26. B. 1. from which the equality of the two triangles does not immediately follow, becauſe to prove that, the 4. of B. 1. muſt likewiſe be made uſe of, as may be ſeen, in the very ſame caſe, in the 34. Prop. B. 1. it was thought better to make uſe only of the 4. of B. 1.

PROP. XLV. B. I.

The ſtraight line KM is proved to be parallel to FL from the 33. Prop. whereas KH is parallel to FG by conſtruction, and KHM, FGL have been demonſtrated to be ſtraight lines. a Corollary is added from Commandine, as being often uſed.

PROP. XIII. B. II.

Book II.

IN this Propoſition only acute angled triangles are mentioned, whereas it holds true of every triangle, and the Demonſtrations of the caſes omitted, are added; Commandine and Clavius have likewiſe given their Demonſtrations of theſe caſes.

PROP. XIV. B. II.

In the Demonſtration of this, ſome Greek Editor has ignorantly inſerted the words, " but if not, one of the two BE, ED is the

Book II. « greater; let BE be the greater and produce it to F," as if it was of any conſequence whether the greater or leſſer be produced. therefore inſtead of theſe words, there ought to be read only, « but if not, produce BE to F."

Book III.

PROP. I. B. III.

SEVERAL Authors, eſpecially among the modern Mathematicians and Logicians, inveigh too ſeverely againſt indirect, or Apagogic Demonſtrations, and ſometimes ignorantly enough; not being aware that there are ſome things that cannot be demonſtrated any other way. of this the preſent Propoſition is a very clear inſtance, as no direct demonſtration can be given of it. becauſe, beſides the Definition of a circle, there is no principle or property relating to a circle antecedent to this Problem, from which either a direct or indirect Demonſtration can be deduced. wherefore it is neceſſary that the point found by the conſtruction of the Problem be proved to be the center of the circle, by the help of this Definition, and ſome of the preceding propoſitions. and becauſe in the Demonſtration, this Propoſition muſt be brought in, viz. ſtraight lines from the center of a circle to the circumference are equal, and that the point found by the conſtruction cannot be aſſumed as the center, for this is the thing to be demonſtrated; it is manifeſt ſome other point muſt be aſſumed as the center; and if from this aſſumption an abſurdity follows, as Euclid demonſtrates there muſt; then it is not true that the point aſſumed is the center; and as any point whatever was aſſumed, it follows that no point, except that found by the conſtruction can be the center. from which the neceſſity of an indirect Demonſtration in this caſe is evident.

PROP. XIII. B. III.

As it is much eaſier to imagine that two circles may touch one another within in more points than one, upon the ſame ſide, than upon oppoſite ſides; the figure of that caſe ought not to have been omitted; but the conſtruction in the Greek text would not have ſuited with this figure ſo well, becauſe the centers of the circles muſt have been placed near to the circumferences. on which account another conſtruction and demonſtration is given which is the ſame with the ſecond part of that which Campanus has tranſlated.

from the Arabic, where without any reaſon the Demonſtration is divided into two parts.

PROP. XV. B. III.

The converſe of the ſecond part of this Propoſition is wanting, tho' in the preceding, the converſe is added, in a like caſe, both in the Enuntiation and Demonſtration; and it is now added in this. beſides in the Demonſtration of the firſt part of this 15th the diameter AD (ſee Commandine's figure) is proved to be greater than the ſtraight line BC by means of another ſtraight line MN; whereas it may be better done without it. on which accounts we have given a different Demonſtration, like to that which Euclid gives in the preceding 14th, and to that which Theodoſius gives, in Prop. 6. B. 1. of his Spherics, in this very affair.

PROP. XVI. B. III.

In this we have not followed the Greek, nor the Latin tranſlation literally, but have given what is plainly the meaning of the Propoſition, without mentioning the angle of the ſemicircle, or that which ſome call the cornicular angle which they conceive to be made by the circumference and the ſtraight line which is at right angles to the diameter, at its extremity; which angles have furniſhed matter of great debate between ſome of the modern Geometers, and given occaſion of deducing ſtrange conſequences from them, which are quite avoided by the manner in which we have expreſſed the Propoſition. and in like manner we have given the true meaning of Prop. 31. B. 3. without mentioning the angles of the greater or leſſer ſegments. theſe paſſages Vieta with good reaſon ſuſpects to be adulterated, in the 386. page of his Oper. Math.

PROP. XX. B. III.

The firſt words of the ſecond part of this Demonſtration, "κεκλάσθω δὴ πάλιν" are wrong tranſlated by Mr. Briggs and Dr. Gregory "Rurſus inclinetur," for the tranſlation ought to be "Rurſus inflectatur" as Commandine has it. a ſtraight line is ſaid to be inflected either to a ſtraight, or curve line, when a ſtraight line is drawn to this line from a point, and from the point in which it meets it, a ſtraight line making an angle with the former is drawn to another point, as is evident from the 90. Prop. of Euclid's Data; for thus the whole line betwixt the firſt and laſt points, is inflected or broken at the point of inflexion where the two ſtraight lines meet. And in the like ſenſe two ſtraight lines are ſaid to be inflected from two points to a third point, when they make an angle at this point; as may be ſeen in the deſcription given by Pappus Alexandrinus of Apollonius Books de Locis planis, in the Preface to his 7. Book. we have made the expreſſion fuller from the 90. Prop. of the Data.

PROP. XXI. B. III.

There are two caſes of this Propoſition, the ſecond of which, viz. when the angles are in a ſegment not greater than a ſemicircle, is wanting in the Greek. and of this a more ſimple Demonſtration is given than that which is in Commandine, as being derived only from the firſt caſe, without the help of triangles.

PROP. XXIII. and XXIV. B. III.

In Propoſition 24. it is demonſtrated that the ſegment AEB muſt coincide with the ſegment CFD (ſee Commandine's figure) and that it cannot fall otherwiſe, as CGD, ſo as to cut the other circle in a third point G, from this, that if it did, a circle could cut another in more points than two. but this ought to have been proved to be impoſſible in the 23. Prop. as well as that one of the ſegments cannot fall within the other. this part then is left out in the 24. and put in its proper place the 23d Propoſition.

PROP. XXV. B. III.

This Propoſition is divided into three caſes, of which two have the ſame conſtruction and demonſtration; therefore it is now divided only into two caſes.

PROP. XXXIII. B. III.

This alſo in the Greek is divided into three caſes, of which two, viz. one, in which the given angle is acute, and the other in which it is obtuſe, have exactly the ſame conſtruction and demonſtration; on which account the demonſtration of the laſt caſe is left out as quite ſuperfluous, and the addition of ſome unſkilful Editor; beſides the demonſtration of the caſe when the angle given is a right angle, is done a round about way, and is therefore changed to a more ſimple one, as was done by Clavius.

PROP. XXXV. B. III.

As the 25. and 33. Propoſitions are divided into more caſes, ſo this 35. is divided into fewer caſes than are neceſſary. Nor can it be ſuppoſed that Euclid omitted them becauſe they are eaſy; as he has given the caſe which by far is the eaſieſt of them all, viz. that in which both the ſtraight lines paſs thro' the center. and in the following Propoſition he ſeparately demonſtrates the caſe in which the ſtraight line paſſes thro' the center, and that in which it does not paſs thro' the center. ſo that it ſeems Theon, or ſome other, has thought them too long to inſert. but caſes that require different demonſtrations, ſhould not be left out in the Elements, as was before taken notice of. theſe caſes are in the tranſlation from the Arabic; and are now put into the Text.

PROP. XXXVII. B. III.

At the end of this the words "in the ſame manner it may be "demonſtrated, if the center be in AC" are left out as the addition of ſome ignorant Editor.

Book IV.

DEFINITIONS OF BOOK IV.

WHEN a point is in a ſtraight, or any other line, this point is by the Greek Geometers ſaid ἅπτεσθαι, to be upon, or in that line. and when a ſtraight line or circle meets a circle any way, the one is ſaid ἅπτεσθαι to meet the other. but when a ſtraight line or circle meets a circle ſo as not to cut it, it is ſaid ἐφάπτεσθαι, to touch the circle; and theſe two terms are never promiſcuouſly uſed by them. therefore in the 5. Definition of B. 4. the compound ἐφάπτηται muſt be read, inſtead of the ſimple ἅπτηται. and in the 1, 2, 3. and 6. Definitions in Commandine's tranſlation "tangit" muſt be read inſtead of "contingit." and in the 2. and 3. Definitions of Book 3. the ſame change muſt be made. but in the Greek text of Propoſitions 11, 12, 13, 18, 19. B. 3. the compound verb is to be put for the ſimple.

PROP. IV. B. IV.

In this, as alſo in the 8. and 13. Propoſitions of this Book, it is demonſtrated indirectly that the circle touches a ſtraight line;

Book IV. whereas in the 17. 33. and 37. Propoſitions of Book 3. the ſame thing is directly demonſtrated. and this way we have choſen to uſe in the Propoſitions of this Book, as it is ſhorter.

PROP. V. B. IV.

The Demonſtration of this has been ſpoiled by ſome unſkilful hand. for he does not demonſtrate, as is neceſſary, that the two ſtraight lines which biſect the ſides of the triangle at right angles, muſt meet one another; and, without any reaſon, he divides the Propoſition into three caſes, whereas one and the ſame conſtruction and demonſtration ſerves for them all, as Campanus has obſerved; which uſeleſs repetitions are now left out. the Greek text alſo in the Corollary is manifeſtly vitiated, where mention is made of a given angle, tho' there neither is, nor can be any thing in the Propoſition relating to a given angle.

PROP. XV. and XVI. B. IV.

In the Corollary of the firſt of theſe, the words equilateral and equiangular are wanting in the Greek. and in Prop. 16. inſtead of the circle ABCD ought to be read the circumference ABCD, where mention is made of its containing fifteen equal parts.

Book V.

DEF. III. B. V.

MANY of the modern Mathematicians reject this Definition. the very learned Dr. Barrow has explained it at large at the end of his third Lecture of the year 1666, in which alſo he anſwers the objections made againſt it as well as the ſubject would allow. and at the end gives his opinion upon the whole, as follows.

" I ſhall only add, that the Author had, perhaps, no other De-
" ſign in making this Definition, than (that he might more fully
" explain and embelliſh his ſubject) to give a general and ſummary
" idea of ratio to beginners, by premiſing this Metaphyſical Defi-
" nition, to the more accurate Definitions of ratios that are the
" ſame to one another, or one of which is greater, or leſs than the
" other. I call it a Metaphyſical, for it is not properly a Mathema-
" tical Definition, ſince nothing in Mathematics depends on it, or
" is deduced, nor, as I judge, can be deduced from it. and the
" Definition of Analogy, which follows, viz. Analogy is the ſimi-

"litude of ratios, is of the ſame kind, and can ſerve for no purpoſe in Mathematics, but only to give beginners ſome general tho' groſs and confuſed notion of Analogy. but the whole of the doctrine of Ratios, and the whole of Mathematics depend upon the accurate Mathematical Definitions which follow this. to theſe we ought principally to attend, as the doctrine of Ratios is more perfectly explained by them; this third, and others like it, may be entirely ſpared without any loſs to Geometry. as we ſee in the 7. Book of the Elements, where the proportion of numbers to one another is defined, and treated of, yet without giving any Definition of the ratio of numbers; tho' ſuch a Definition was as neceſſary and uſeful to be given in that Book, as in this. but indeed there is ſcarce any need of it in either of them. tho' I think that a thing of ſo general and abſtracted a nature, and thereby the more difficult to be conceived, and explained, cannot be more commodiouſly defined, than as the Author has done. upon which account I thought fit to explain it at large, and defend it againſt the captious objections of thoſe who attack it." to this citation from Dr. Barrow I have nothing to add, except that I fully believe the 3. and 8. Definitions are not Euclid's, but added by ſome unſkilful Editor.

DEF. XI. B. V.

It was neceſſary to add the word "continual" before "proportionals" in this Definition; and thus it is cited in the 33. Prop. of Book 11.

After this Definition ought to have followed the Definition of Compound ratio, as this was the proper place for it; Duplicate and Triplicate ratio being ſpecies of Compound ratio. But Theon has made it the 5. Def. of B. 6. where he gives an abſurd and entirely uſeleſs Definition of Compound ratio. for this reaſon we have placed another Definition of it betwixt the 11. and 12. of this Book, which, no doubt, Euclid gave; for he cites it expreſsly in Prop. 23. B. 6. and which Clavius, Herigon and Barrow have likewiſe given, but they retain alſo Theon's, which they ought to have left out of the Elements.

DEF. XIII. B. V.

This and the reſt of the Definitions following, contain the explication of ſome terms which are uſed in the 5. and following

Books; which, except a few, are eaſily enough underſtood from the Propoſitions of this Book where they are firſt mentioned. they ſeem to have been added by Theon or ſome other. However it be, they are explained ſomething more diſtinctly for the ſake of learners.

PROP. IV. B. V.

In the conſtruction, preceding the demonſtration of this, the words ἃ ἔτυχε, any whatever, are twice wanting in the Greek, as alſo in the Latin tranſlations; and are now added, as being wholly neceſſary.

Ibid. in the demonſtration; in the Greek, and in the Latin tranſlation of Commandine, and in that of Mr. Henry Briggs, which was publiſhed at London in 1620, together with the Greek text of the firſt ſix Books, which tranſlation in this place is followed by Dr. Gregory in his edition of Euclid, there is this ſentence following, viz. “and of A and C have been taken equimultiples K, L; “and of B and D, any equimultiples whatever (ἃ ἔτυχε) M, N;” which is not true. the words “any whatever” ought to be left out. and it is ſtrange that neither Mr. Briggs, who did right to leave out theſe words in one place of Prop. 13. of this Book, nor Dr. Gregory who changed them into the word “ſome” in three places, and left them out in a fourth of that ſame Prop. 13. did not alſo leave them out in this place of Prop. 4. and in the ſecond of the two places where they occur in Prop. 17. of this Book, in neither of which they can ſtand conſiſtent with truth. and in none of all theſe places, even in thoſe which they corrected in their Latin tranſlation, have they cancelled the words ἃ ἔτυχε in the Greek text, as they ought to have done.

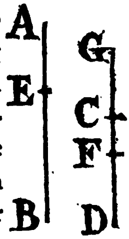
The ſame words ἃ ἔτυχε are found in four places of Prop. 11. of this Book, in the firſt and laſt of which, they are neceſſary, but in the ſecond and third, tho' they are true, they are quite ſuperfluous; as they likewiſe are in the ſecond of the two places in which they are found in the 12. Prop. and in the like places of Prop. 22, 23. of this Book. but are wanting in the laſt place of Prop. 23. as alſo in Prop. 25. B. 11.

COR. PROP. IV. B. V.

This Corollary has been unſkilfully annexed to this Propoſition, and has been made inſtead of the legitimate demonſtration which without doubt Theon, or ſome other Editor has taken away, not from this, but from its proper place in this Book. the Author of it deſigned to demonſtrate that if four magnitudes E, G, F, H be proportionals, they are alſo proportionals inverſely; that is, G is to E, as H to F: which is true, but the demonſtration of it does not in the leaſt depend upon this 4. Prop. or its demonſtration. for when he ſays "becauſe it is demonſtrated that if K be greater "than M, L is greater than N," &c. this is indeed ſhewn in the demonſtration of the 4. Prop. but not from this that E, G, F, H are proportionals, for this laſt is the concluſion of the Propoſition. Wherefore theſe words "becauſe it is demonſtrated," &c. are wholly foreign to his deſign. and he ſhould have proved that if K be greater than M, L is greater than N, from this, that E, G, F, H are proportionals, and from the 5. Def. of this Book, which he has not; but is done in Propoſition B, which we have given, in its proper place, inſtead of this Corollary. and another Corollary is placed after the 4. Prop. which is often of uſe, and is neceſſary to the Demonſtration of Prop. 18. of this Book.

PROP. V. B. V.

In the conſtruction which precedes the demonſtration of this Propoſition, it is required that EB may be the ſame multiple of CG, that AE is of CF; that is, that EB be divided into as many equal parts, as there are parts in AE equal to CF. from which it is evident that this conſtruction is not Euclid's. for he does not ſhew the way of dividing ſtraight lines, and far leſs other magnitudes, into any number of equal parts, until the 9. Propoſition of B. 6. and he never requires any thing to be done in the conſtruction, of which he had not before given the method of doing. for this reaſon we have changed the conſtruction to one which without doubt is Euclid's, in which nothing is required but to add a magnitude to itſelf a certain number of times. and this is to be found in the tranſlation from the Arabic, tho' the enunciation of the Propoſition and the demonſtration are there very much ſpoiled. Jacobus Peletarius who was the firſt, as far as I know, who took notice of this error, gives alſo the right conſtruction in his edition of Euclid, after he had given the other which he blames. he ſays he would not leave it out, becauſe it was fine, and might ſharpen one's genius to invent others



like it; whereas there is not the leaſt difference between the two demonſtrations, except a ſingle word in the conſtruction, which very probably has been owing to an unſkilful Librarian. Clavius likewiſe gives both the ways, but neither he nor Peletarius takes notice of the reaſon why one is preferable to the other.

PROP. VI. B. V.

There are two caſes of this Propoſition, of which only the firſt and ſimpleſt is demonſtrated in the Greek. and it is probable Theon thought it was ſufficient to give this one, ſince he was to make uſe of neither of them in his mutilated edition of the 5th Book; and he might as well have left out the other, as alſo the 5. Propoſition for the ſame reaſon. the demonſtration of the other caſe is now added, becauſe both of them, as alſo the 5. Propoſition, are neceſſary to the demonſtration of the 18. Prop. of this Book. the tranſlation from the Arabic gives both caſes briefly.

PROP. A. B. V.

This Propoſition is frequently uſed by Geometers, and it is neceſſary in the 25. Prop. of this Book, 31. of the 6. and 34. of the 11. and 15. of the 12. Book. it ſeems to have been taken out of the Elements by Theon, becauſe it appeared evident enough to him, and others who ſubſtitute the confuſed and indiſtinct idea the vulgar have of proportionals, in place of that accurate idea which is to be got from the 5. Def. of this Book. Nor can there be any doubt that Eudoxus or Euclid gave it a place in the Elements, when we ſee the 7. and 9. of the ſame Book demonſtrated, tho' they are quite as eaſy and evident as this. Alphonſus Borellus takes occaſion from this Propoſition to cenſure the 5. Definition of this Book very ſeverely, but moſt unjuſtly. in page 126. of his Euclid reſtored, printed at Piſa in 1658. he ſays, "Nor can even this leaſt degree "of knowledge be obtained from the foreſaid property," viz. that which is contained in 5. Def. 5. "That if four magnitudes be "proportionals, the third muſt neceſſarily be greater than the "fourth, when the firſt is greater than the ſecond; as Clavius ac"knowledges in the 16. Prop. of the 5. Book of the Elements." But tho' Clavius makes no ſuch acknowledgement expreſsly, he has given Borellus a handle to ſay this of him, becauſe when Clavius in the above-cited place cenſures Commandine, and that very juſtly, for demonſtrating this Propoſition by help of the 16. of the

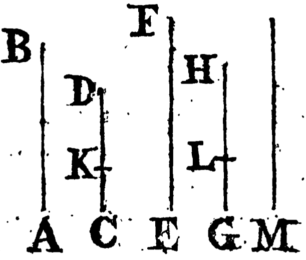
5. yet he himſelf gives no demonſtration of it, but thinks it plain from the nature of Proportionals, as he writes in the end of the 14. and 16. Prop. B. 5. of his edition, and is followed by Herigon in Schol. 1. Prop. 14. B. 5. as if there was any nature of Proportionals antecedent to that which is to be derived and underſtood from the Definition of them. and indeed, tho' it is very eaſy to give a right demonſtration of it, no body, as far as I know, has given one, except the learned Dr. Barrow, who, in anſwer to Borellus's objection, demonſtrates it indirectly, but very briefly and clearly from the 5. Definition, in the 322 page of his Lect. Mathem. from which Definition it may alſo eaſily be demonſtrated directly. on which account we have placed it next to the Propoſitions concerning equimultiples.

PROP. B. B. V.

This alſo is eaſily deduced from the 5. Def. B. 5. and therefore is placed next to the other, for it was very ignorantly made a Corollary from the 4. Prop. of this Book. See the note on that Corollary.

PROP. C. B. V.

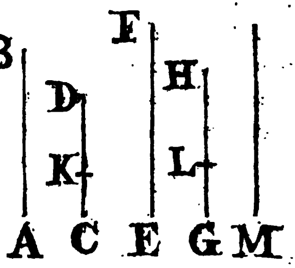
This is frequently made uſe of by Geometers, and is neceſſary to the 5. and 6. Propoſitions of the 10. Book. Clavius in his Notes ſubjoined to the 8. Def. of Book 5. demonſtrates it only in numbers, by help of ſome of the Propoſitions of the 7. Book, in order to demonſtrate the property contained in the 5. Definition of the 5. Book, when applied to numbers, from the property of Proportionals contained in the 20. Def. of the 7. Book. and moſt of the Commentators judge it difficult to prove that four magnitudes which are proportionals according to the 20. Def. of 7. B. are alſo proportionals according to the 5. Def. of 5. Book. but this is eaſily made out, as follows.



Firſt, If A, B, C, D be four magnitudes, ſuch that A is the ſame multiple, or the ſame part of B, which C is of D; A, B, C, D are proportionals. this is demonſtrated in Propoſition C.

Secondly, If AB contain the ſame parts of CD that EF does of GH; in this caſe likewiſe AB is to CD, as EF to GH.

Let CK be a part of CD, and GL the ſame part of GH; and let AB be the ſame multiple of CK, that EF is of GL. therefore by Prop. C. of 5. Book, AB is to CK, as EF to GL. and CD, GH are equimultiples of CK, GL the ſecond and fourth; wherefore by Cor. Prop. 4. B. 5. AB is to CD, as EF to GH.



And if four magnitudes be proportionals according to the 5. Def. of B. 5. they are alſo proportionals, according to the 20. Def. of. B. 7.

Firſt, If A be to B, as C to D; then if A be any multiple or part of B, C is the ſame multiple or part of D, by Prop. D of B. 5.

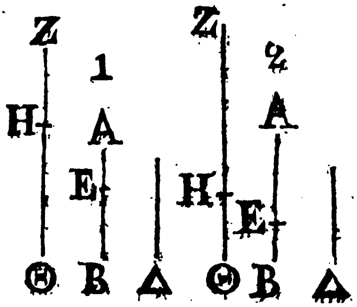
Next, If AB be to CD, as EF to GH; then if AB contains any parts of CD, EF contains the ſame parts of GH. for let CK be a part of CD, and GL the ſame part of GH, and let AB be a multiple of CK; EF is the ſame multiple of GL. Take M the ſame multiple of GL that AB is of CK; therefore by Prop. C. of B. 5. AB is to CK, as M to GL; and CD, GH are equimultiples of CK, GL; wherefore by Cor. Prop. 4. B. 5. AB is to CD, as M to GH. and, by the Hypotheſis, AB is to CD, as EF to GH; therefore M is equal to EF by Prop. 9. B. 5. and conſequently EF is the ſame multiple of GL that AB is of CK.

PROP. D. B. V.

This is not unfrequently uſed in the demonſtration of other Propoſitions, and is neceſſary in that of Prop. 9. B. 6. it ſeems Theon has left it out for the reaſon mentioned in the Notes at Prop. A.

PROP. VIII. B. V.

In the demonſtration of this, as it is now in the Greek, there are two caſes, (ſee the demonſtration in Hervagius, or Dr. Gregory's edition) of which the firſt is that in which AE is leſs than EB; and in this, it neceſſarily follows that HΘ the multiple of EB is greater than ZH the ſame multiple of AE, which laſt multiple, by the conſtruction, is greater than Δ; whence alſo HΘ muſt be greater than Δ. but in the ſecond caſe, viz. that in which EB is leſs than AE, tho' ZH be greater than Δ, yet HΘ may be leſs than the ſame Δ; ſo that there cannot be taken a multiple of Δ which is the firſt that is greater than K, or HΘ, becauſe Δ itſelf is greater than it. upon this account, the Author of this demonſtration found it neceſſary to change one part of the conſtruction that was made uſe of in the firſt caſe. but he has, without any neceſſity, changed alſo another part of it, viz. when he orders to take N that multiple of Δ which is the firſt that is greater than ZH; for he might have taken that multiple of Δ which is the firſt that is greater than HΘ, or K, as was done in the firſt caſe. he likewiſe brings in this K into the demonſtration of both caſes, without any reaſon, for it ſerves to no purpoſe but to lengthen the demonſtration. There is alſo a third caſe, which is not mentioned in this demonſtration, viz. that in which AE in the firſt, or EB in the ſecond of the two other caſes, is greater than D; and in this any equimultiples, as the doubles, of AE, EB are to be taken, as is done in this Edition, where all the caſes are at once demonſtrated. and from this it is plain that Theon, or ſome other unſkilful Editor has vitiated this Propoſition.



PROP. IX. B. V.

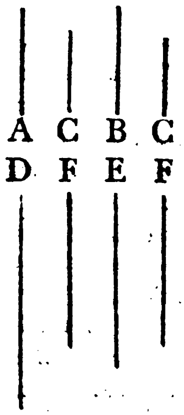
Of this there is given a more explicit demonſtration than that which is now in the Elements.

PROP. X. B. V.

It was neceſſary to give another demonſtration of this Propoſition, becauſe that which is in the Greek, and Latin, or other editions, is not legitimate. for the words *greater*, *the ſame* or *equal*, *leſſer* have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5. and 7. Definitions of B. 5. by the help of theſe let us examine the demonſtration of the 10. Prop. which proceeds thus. “Let A have to C a greater ratio, than B to C. I ſay “that A is greater than B. for if it is not greater, it is either equal, “or leſs. but A cannot be equal to B, becauſe then each of them “would have the ſame ratio to C; but they have not. therefore “A is not equal to B.” the force of which reaſoning is this, if A had to C, the ſame ratio that B has to C, then if any equimultiples whatever of A and B be taken, and any multiple whatever of C; if the multiple of A be greater than the multiple of C, then, by the 5. Def. of B. 5. the multiple of B is alſo greater than that of C. but from the Hypotheſis that A has a greater ratio to C, than B has to C, there muſt, by the 7. Def. of B. 5. be certain equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than the ſame multiple of C. and this Propoſition directly contradicts the preceding; wherefore A is not equal to B. the demonſtration of the 10. Propoſition goes on thus, “but neither is A leſs “than B, becauſe then A would have a leſs ratio to C, than B has “to it. but it has not a leſs ratio, therefore A is not leſs than B,” &c. here it is ſaid that “A would have a leſs ratio to C, than B has “to C,” or, which is the ſame thing, that B would have a greater ratio to C, than A to C; that is, by 7. Def. B. 5. there muſt be ſome equimultiples of B and A, and ſome multiple of C ſuch, that the multiple of B is greater than the multiple of C, but the multiple of A is not greater than it. and it ought to have been proved that this can never happen if the ratio of A to C, be greater than the ratio of B to C; that is, it ſhould have been proved that in this caſe the multiple of A is always greater than the multiple of C, whenever the multiple of B is greater than the multiple of C; for when this is demonſtrated it will be evident that B cannot have a greater ratio to C, than A has to C, or, which is the ſame thing, that A cannot have a leſs ratio to C, than B has to C. but this is not at all proved in the 10. Propoſition; but if the 10. were once demonſtrated it would immediately follow from it; but cannot without it be eaſily demonſtrated, as he that tries to do it will find. wherefore the 10. Propoſition is not ſufficiently demonſtrated. and it ſeems that he who has given the demonſtration of the 10. Propoſition as we now have it, inſtead of that which Eudoxus or Euclid had given, has been deceived in applying what was manifeſt when underſtood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and leſs than another. That thoſe things which are equal to the ſame are equal to one another, is a moſt evident Axiom when underſtood of magnitudes, yet Euclid does not make uſe of it to infer that thoſe ratios which are the ſame to the ſame ratio, are the ſame to one another; but explicitely demonſtrates this in Prop. 11. of B. 5. the demonſtration we have given of the 10.

Prop. is no doubt the ſame with that of Eudoxus or Euclid, as it is immediately and directly derived from the Definition of a greater ratio, viz. the 7. of the 5.

The above-mentioned Propoſition, viz. If A have to C a greater ratio than B to C, and if of A and B there be taken certain equimultiples, and ſome multiple of C; then if the multiple of B be greater than the multiple of C, the multiple of A is alſo greater than the ſame, is thus demonſtrated.



Let D, E be equimultiples of A, B, and F a multiple of C, ſuch, that E the multiple of B is greater than F; D the multiple of A is alſo greater than F.

Becauſe A has a greater ratio to C, than B to C, A is greater than B, by the 10. Prop. B. 5. therefore D the multiple of A is greater than E the ſame multiple of B. and E is greater than F; much more therefore D is greater than F.

PROP. XIII. B. V.

In Commandine's, Briggs's and Gregory's Tranſlations, at the beginning of this demonſtration, it is ſaid, "And the multiple of C "is greater than the multiple of D; but the multiple of E is not "greater than the multiple of F," which words are a literal tranſlation from the Greek. but the ſenſe evidently requires that it be read, "ſo that the multiple of C be greater than the multiple of D; "but the multiple of E be not greater than the multiple of F." and thus this place was reſtored to the true reading in the firſt editions of Commandine's Euclid printed in 8vo. at Oxford; but in the later editions, at leaſt in that of 1747, the error of the Greek text was kept in.

There is a Corollary added to Prop. 13. as it is neceſſary to the 20. and 21. Prop. of this Book, and is as uſeful as the Propoſition.

PROP. XIV. B. V.

The two caſes of this which are not in the Greek are added; the demonſtration of them not being exactly the ſame with that of the firſt caſe.

PROP. XVII. B. V.

The order of the words in a clauſe of this is changed to one more natural. as was alſo done in Prop. 11.

PROP. XVIII. B. V.

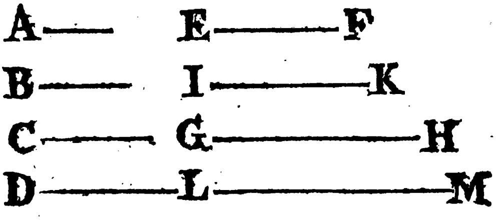
The demonſtration of this is none of Euclid's, nor is it legitimate. for it depends upon this Hypotheſis, that to any three magnitudes, two of which, at leaſt, are of the ſame kind, there may be a fourth proportional; which if not proved, the Demonſtration now in the text, is of no force. but this is aſſumed without any proof, nor can it, as far as I am able to diſcern, be demonſtrated by the Propoſitions preceding this; ſo far is it from deſerving to be reckoned an Axiom, as Clavius, after other Commentators, would have it, at the end of the Definitions of the 5. Book. Euclid does not demonſtrate it, nor does he ſhew how to find the fourth proportional, before the 12. Prop. of the 6. Book. and he never aſſumes any thing in the demonſtration of a Propoſition, which he had not before demonſtrated; at leaſt, he aſſumes nothing the exiſtence of which is not evidently poſſible; for a certain concluſion can never be deduced by the means of an uncertain Propoſition. upon this account we have given a legitimate Demonſtration of this Propoſition inſtead of that in the Greek and other editions, which very probably Theon, at leaſt ſome other has put in the place of Euclid's, becauſe he thought it too prolix. and as the 17. Prop. of which this 18. is the converſe, is demonſtrated by help of the 1. and 2. Propoſitions of this Book, ſo in the demonſtration now given of the 18th, the 5. Prop. and both caſes of the 6. are neceſſary, and theſe two Propoſitions are the converſes of the 1. and 2. Now the 5. and 6. do not enter into the demonſtration of any Propoſition in this Book as we now have it, nor can they be of uſe in any Propoſition of the Elements, except in this 18. and this is a manifeſt proof that Euclid made uſe of them in his demonſtration of it, and that the demonſtration now given, which is exactly the converſe of that of the 17. as it ought to be, differs nothing from that of Eudoxus or Euclid. for the 5. and 6. have undoubtedly been put into the 5. Book for the ſake of ſome Propoſitions in it, as all the other Propoſitions about equimultiples have been.

Hieronymus Saccherius in his Book named Euclides ab omni naevo vindicatus, printed at Milan Ann. 1733 in 4to, acknowledges this blemiſh in the demonſtration of the 18. and that he may remove it, and render the demonſtration we now have of it legitimate, he endeavours to demonſtrate the following Propoſition, which is in page 115 of his Book, viz.

“ Let A, B, C, D be four magnitudes, of which the two firſt
“ are of one kind, and alſo the two others either of the ſame kind
“ with the two firſt, or of ſome other the ſame kind with one
“ another. I ſay the ratio of the third C to the fourth D, is
“ either equal to, or greater, or leſs than the ratio of the firſt A
“ to the ſecond B.”

And after two Propoſitions premiſed as Lemmas, he proceeds thus.

“ Either among all the poſſible equimultiples of the firſt A, and
“ of the third C, and, at the ſame time among all the poſſible equi-
“ multiples of the ſecond B, and of the fourth D, there can be found
“ ſome one multiple EF of the firſt A, and one IK of the ſecond B,
“ that are equal to one another; and alſo (in the ſame caſe) ſome
“ one multiple GH of the third C equal to LM the multiple of the
“ fourth D. or ſuch equality is no where to be found. If the firſt
“ caſe happen, [i. e.
“ if ſuch equality
“ is to be found,]
“ it is manifeſt from
“ what is before de-
“ monſtrated, that
“ A is to B, as C
“ to D. but if ſuch
“ ſimultaneous equality be not to be found upon both ſides, it will
“ be found either upon one ſide, as upon the ſide of A [and B;]
“ or it will be found upon neither ſide; if the firſt happen; there-
“ fore (from Euclid's Definition of greater and leſſer ratio fore-
“ going) A has to B, a greater or leſs ratio than C to D; accor-
“ ding as GH the multiple of the third C is leſs, or greater than
“ LM the multiple of the fourth D. but if the ſecond caſe hap-
“ pen; therefore upon the one ſide, as upon the ſide of A the firſt
“ and B the ſecond, it may happen that the multiple EF, [viz. of the
“ firſt] may be leſs than IK the multiple of the ſecond, while on
“ the contrary, upon the other ſide, [viz. of C and D] the multiple
“ GH [of the third C] is greater than the other multiple LM [of
“ the fourth D.] and then (from the ſame Definition of Euclid) the



"ratio of the firſt A to the ſecond B, is leſs than the ratio of "the third C to the fourth D; or on the contrary.

"Therefore the Axiom, [i. e. the Propoſition before ſet down,] "remains demonſtrated," &c.

Not in the leaſt; but it remains ſtill undemonſtrated: for what he ſays may happen, may in innumerable caſes never happen; and therefore his demonſtration does not hold. for example, if A be the ſide and B the diameter of a ſquare; and C the ſide and D the diameter of another ſquare; there can in no caſe be any multiple of A equal to any of B; nor any one of C equal to one of D, as is well known; and yet it can never happen that when any multiple of A is greater or leſs than a multiple of B, the multiple of C can, upon the contrary, be leſs or greater than the multiple of D, viz. taking equimultiples of A and C, and equimultiples of B and D. for A, B, C, D are proportionals, and ſo if the multiple of A be greater, &c. than that of B, ſo muſt that of C be greater &c. than that of D, by 5. Def. B. 5.

The ſame objection holds good againſt the Demonſtration which ſome give of the 1. Prop. of the 6. Book, which we have made againſt this of the 18. Propoſition, becauſe it depends upon the ſame inſufficient foundation with the other.

PROP. XIX. B. V.

A Corollary is added to this, which is as frequently uſed as the Propoſition itſelf. the Corollary which is ſubjoined to it in the Greek, plainly ſhews that the 5. Book has been vitiated by Editors who were not Geometers. for the converſion of ratios does not depend upon this 19. and the Demonſtration which ſeveral of the Commentators on Euclid give of Converſion, is not legitimate, as Clavius has rightly obſerved, who has given a good Demonſtration of it which we have put in Propoſition E; but he makes it a Corollary from the 19. and begins it with the words, "Hence "it eaſily follows," tho' it does not at all follow from it.

PROP. XX, XXI, XXII, XXIII, XXIV. B. V.

The Demonſtrations of the 20. and 21. Propoſitions are ſhorter than thoſe Euclid gives of eaſier Propoſitions, either in the preceding, or following Books. wherefore it was proper to make them more explicit. and the 22. and 23. Propoſitions are, as they ought to be, extended to any number of magnitudes. and in like manner may the 24. be, as is taken notice of in a Corollary; and another Corollary is added, as uſeful as the Propoſition. and the words "any whatever." are ſupplied near the end of Prop. 23. which are wanting in the Greek text, and the tranſlations from it.

In a paper writ by Philippus Naudaeus, and publiſhed, after his death, in the Hiſtory of the Royal Academy of Sciences of Berlin, Ann. 1745. page 50. the 23. Prop. of the 5. Book, is cenſured as being obſcurely enuntiated, and, becauſe of this, prolixly demonſtrated. the Enuntiation there given is not Euclid's but Tacquet's, as he acknowledges, which, tho' not ſo well expreſſed, is, upon the matter, the ſame with that which is now in the Elements. Nor is there any thing obſcure in it, tho' the Author of the paper has ſet down the proportionals in a diſadvantageous order, by which it appears to be obſcure. but no doubt Euclid enuntiated this 23. as well as the 22. ſo as to extend it to any number of magnitudes, which taken two and two, are proportionals, and not of ſix only; and to this general caſe the Enuntiation which Naudaeus gives, cannot be well applied.

The Demonſtration which is given of this 23. in that paper, is quite wrong; becauſe if the proportional magnitudes be plane or ſolid figures, there can no rectangle (which he improperly calls a Product) be conceived to be made by any two of them. and if it ſhould be ſaid, that in this caſe ſtraight lines are to be taken which are proportional to the figures, the Demonſtration would this way become much longer than Euclid's. but even tho' his Demonſtration had been right, who does not ſee that it could not be made uſe of in the 5. Book?

PROP. F, G, H, K. B. V.

Theſe Propoſitions are annexed to the 5. Book, becauſe they are frequently made uſe of by both antient and modern Geometers. and in many caſes Compound ratios cannot be brought into Demonſtrations, without making uſe of them.

Whoever deſires to ſee the doctrine of Ratios delivered in this 5. Book, ſolidly defended, and the arguments brought againſt it by And. Tacquet, Alph. Borellus and others, fully refuted, may read Dr. Barrow's Mathematical Lectures, viz. the 7. and 8. of the year 1666.

The 5. Book being thus corrected, I moſt readily agree to what the learned Dr. Barrow ſays*, "That there is nothing in the whole *Page 336.

Book V. "body of the Elements, of a more ſubtile invention, nothing more "ſolidly eſtabliſhed and more accurately handled, than the doc-"trine of Proportionals." And there is ſome ground to hope that Geometers will think that this could not have been ſaid with as good reaſon, ſince Theon's time till the preſent.

Book VI.

DEF. II. and V. of B. VI.

THE 2. Definition does not ſeem to be Euclid's but ſome unſkilful Editor's. for there is no mention made by Euclid, nor, as far as I know, by any other Geometer, of reciprocal figures. it is obſcurely expreſſed, which made it proper to render it more diſtinct. it would be better to put the following Definition in place of it, viz.

DEF. II.

Two magnitudes are ſaid to be reciprocally proportional to two others, when one of the firſt is to one of the other magnitudes, as the remaining one of the laſt two is to the remaining one of the firſt.

But the 5. Definition, which ſince Theon's time has been kept in the Elements, to the great detriment of learners, is now juſtly thrown out of them, for the reaſons given in the Notes on the 23. Prop. of this Book.

PROP. I. and II. B. VI.

To the firſt of theſe a Corollary is added which is often uſed. and the Enuntiation of the ſecond is made more general.

PROP. III. B. VI.

A ſecond caſe of this, as uſeful as the firſt, is given in Prop. A, viz. the caſe in which the exterior angle of a triangle is biſected by a ſtraight line. the Demonſtration of it is very like to that of the firſt caſe, and upon this account may, probably, have been left out, as alſo the Enuntiation, by ſome unſkilful Editor. at leaſt it is certain that Pappus makes uſe of this caſe, as an Elementary Propoſition, without a Demonſtration of it, in Prop. 39. of his 7. Book of Mathem. Collections.

PROP. VII. B. VI.

To this a caſe is added which occurs not unfrequently in Demonſtrations.

PROP. VIII. B. VI.

It ſeems plain that ſome Editor has changed the Demonſtration that Euclid gave of this Propoſition. for after he has demonſtrated that the triangles are equiangular to one another, he particularly ſhews that their ſides about the equal angles are proportionals, as if this had not been done in the Demonſtration of the 4. Prop. of this Book. this ſuperfluous part is not found in the Tranſlation from the Arabic, and is now left out.

PROP. IX. B. VI.

This is demonſtrated in a particular caſe, viz. that in which the third part of a ſtraight line is required to be cut off; which is not at all like Euclid's manner. beſides, the Author of the Demonſtration, from four magnitudes being proportionals, concludes that the third of them is the ſame multiple of the fourth, which the firſt is of the ſecond; now this is no where demonſtrated in the 5. Book, as we now have it. but the Editor aſſumes it from the confuſed notion which the vulgar have of proportionals. on this account it was neceſſary to give a general and legitimate Demonſtration of this Propoſition.

PROP. XVIII. B. VI.

The Demonſtration of this ſeems to be vitiated. for the Propoſition is demonſtrated only in the caſe of quadrilateral figures, without mentioning how it may be extended to figures of five or more ſides. beſides, from two triangles being equiangular it is inferred that a ſide of the one is to the homologous ſide of the other, as another ſide of the firſt is to the ſide homologous to it of the other, without permutation of the proportionals; which is contrary to Euclid's manner, as is clear from the next Propoſition. and the ſame fault occurs again in the concluſion, where the ſides about the equal angles are not ſhewn to be proportionals; by reaſon of again neglecting permutation. on theſe accounts a Demonſtration is given in Euclid's manner, like to that he makes uſe of in the 20. Prop.

of this Book; and it is extended to five ſided figures, by which it may be ſeen how to extend it to figures of any number of ſides.

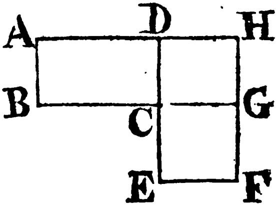
PROP. XXIII. B. VI.

Nothing is uſually reckoned more difficult in the Elements of Geometry by learners, than the doctrine of Compound ratio, which Theon has rendered abſurd and ungeometrical, by ſubſtituting the 5. Definition of the 6. Book, in place of the right Definition which without doubt Eudoxus or Euclid gave, in its proper place, after the Definition of Triplicate ratio, &c. in the 5. Book. Theon's Definition is this; a Ratio is ſaid to be compounded of ratios ὅταν αἱ τῶν λογων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσι τινα. which Commandine thus tranſlates, "quando rationum quantitates "inter ſe multiplicatae aliquam efficiunt rationem;" that is, when the quantities of the ratios being multiplied by one another make a certain ratio. Dr. Wallis tranſlates the word πηλικότητες, "rationum exponentes," the exponents of the ratios. and Dr. Gregory renders the laſt words of the Definition by "illius facit quantitatem," makes the quantity of that ratio. but in whatever ſenſe the "quantities" or "exponents of the ratios," and their "multiplication" be taken, the Definition will be ungeometrical and uſeleſs. for there can be no multiplication but by a number; now the quantity or exponent of a ratio (according as Eutocius in his Comment. on Prop. 4. Book 2. of Arch. de Sph. et Cyl. and the moderns explain that term) is the number which multiplied into the conſequent term of a ratio produces the antecedent, or, which is the ſame thing, the number which ariſes by dividing the antecedent by the conſequent; but there are many ratios ſuch, that no number can ariſe from the diviſion of the antecedent by the conſequent; ex. gr. the ratio which the diameter of a ſquare has to the ſide of it; and the ratio which the circumference of a circle has to its diameter, and ſuch like. Beſides, there is not the leaſt mention made of this Definition in the writings of Euclid, Archimedes, Apollonius, or other antients, tho' they frequently make uſe of Compound ratio. and in this 23. Prop. of the 6. Book, where Compound ratio is firſt mentioned, there is not one word which can relate to this Definition, tho' here, if in any place, it was neceſſary to be brought in; but the right Definition is expreſsly cited in theſe words, "but the ratio of K to "M is compounded of the ratio of K to L, and of the ratio of L "to M." this Definition therefore of Theon is quite uſeleſs and abſurd. for that Theon brought it into the Elements can ſcarce be doubted, as it is to be found in his Commentary upon Ptolomy's Μεγάλη Συντάξις, page 62. where he alſo gives a childiſh explication of it, as agreeing only to ſuch ratios as can be expreſſed by numbers; and from this place the Definition and explication have been exactly copied and prefixed to the Definitions of the 6. Book, as appears from Hervagius's edition. but Zambertus and Commandine in their Latin Tranſlations ſubjoin the ſame to theſe Definitions. Neither Campanus, nor, as it ſeems, the Arabic manuſcripts from which he made his Tranſlation, have this Definition. Clavius in his Obſervations upon it, rightly judges that the Definition of Compound ratio might have been made after the ſame manner in which the Definitions of Duplicate and Triplicate ratio are given, viz. “ that as in ſeveral magnitudes that are continual proportionals, Euclid named the ratio of the firſt to the third, the Duplicate ratio “ of the firſt to the ſecond; and the ratio of the firſt to the fourth, “ the Triplicate ratio of the firſt to the ſecond; that is, the ratio “ compounded of two or three intermediate ratios that are equal “ to one another, and ſo on; ſo in like manner if there be ſeveral “ magnitudes of the ſame kind, following one another, which are “ not continual proportionals, the firſt is ſaid to have to the laſt “ the ratio compounded of all the intermediate ratios,——only for “ this reaſon, that theſe intermediate ratios are interpoſed betwixt “ the two extremes, viz. the firſt and laſt magnitudes; even as in “ the 10. Definition of the 5. Book, the ratio of the firſt to the “ third was called the Duplicate ratio, merely upon account of two “ ratios being interpoſed betwixt the extremes, that are equal to one “ another. ſo that there is no difference betwixt this compounding “ of ratios, and the duplication or triplication of them which are “ defined in the 5. Book, but that in the duplication, triplication, “ &c. of ratios, all the interpoſed ratios are equal to one another; “ whereas in the compounding of ratios, it is not neceſſary that the “ intermediate ratios ſhould be equal to one another.” Alſo Mr. Edmund Scarburgh, in his Engliſh tranſlation of the firſt ſix Books, page 238, 266. expreſsly affirms that the 5. Definition of the 6. Book, is ſuppoſititious, and that the true Definition of Compound ratio is contained in the 10. Definition of the 5. Book, viz. the Definition of Duplicate ratio, or to be underſtood from it, to wit, in the ſame manner as Clavius has explained it in the preceding citation. Yet theſe, and the reſt of the Moderns, do notwithſtanding

Book VI. retain this 5. Def. of the 6. B. and illuſtrate and explain it by long Commentaries, when they ought rather to have taken it quite away from the Elements.

For, by comparing Def. 5. B. 6. with Prop. 5. B. 8. it will clearly appear that this Definition has been put into the Elements in place of the right one which has been taken out of them. becauſe in Prop. 5. B. 8. it is demonſtrated that the plane number of which the ſides are C, D has to the plane number of which the ſides are E, Z (ſee Hervagius's or Gregory's Edition) the ratio which is compounded of the ratios of their ſides; that is, of the ratios of C to E, and D to Z. and by Def. 5. B. 6. and the explication given of it by all the Commentators, the ratio which is compounded of the ratios of C to E, and D to Z, is the ratio of the product made by the multiplication of the antecedents C, D, to the product of the conſequents E, Z, that is the ratio of the plane number of which the ſides are C, D to the plane number of which the ſides are E, Z. wherefore the Propoſition which is the 5. Def. of B. 6. is the very ſame with the 5. Prop. of B. 8. and therefore it ought neceſſarily to be cancelled in one of theſe places; becauſe it is abſurd that the ſame Propoſition ſhould ſtand as a Definition in one place of the Elements, and be demonſtrated in another place of them. Now there is no doubt that Prop. 5. Book. 8. ſhould have a place in the Elements, as the ſame thing is demonſtrated in it concerning plane numbers, which is demonſtrated in Prop. 23. B. 6. of equiangular parallelograms; wherefore Def. 5. B. 6. ought not to be in the Elements. and from this it is evident that this Definition is not Euclid's but Theon's, or ſome other unſkilful Geometer's.

But no body, as far as I know, has hitherto ſhewn the true uſe of Compound ratio, or for what purpoſe it has been introduced into Geometry; for every Propoſition in which Compound ratio is made uſe of, may without it be both enuntiated and demonſtrated. Now the uſe of Compound ratio conſiſts wholly in this, that by means of it, circumlocutions may be avoided, and thereby Propoſitions may be more briefly either enuntiated or demonſtrated, or both may be done; for inſtance, if this 23. Propoſition of the 6. Book were to be enuntiated, without mentioning Compound ratio, it might be done as follows; If two Parallelograms be equiangular, and if as a ſide of the firſt to a ſide of the ſecond, ſo any aſſumed ſtraight line be made to a ſecond ſtraight line; and as the other ſide of the firſt to the other ſide of the ſecond, ſo the ſecond ſtraight line be made to a third: the firſt parallelogram is to the ſecond, as the firſt ſtraight line to the third. and the Demonſtration would be exactly the ſame as we now have it. but the antient Geometers, when they obſerved this Enuntiation could be made ſhorter, by giving a name to the ratio which the firſt ſtraight line has to the laſt, by which name the intermediate ratios might likewiſe be ſignified, of the firſt to the ſecond, and of the ſecond to the third, and ſo on if there were more of them, they called this ratio of the firſt to the laſt, the ratio compounded of the ratios of the firſt to the ſecond, and of the ſecond to the third ſtraight line; that is, in the preſent example, of the ratios which are the ſame with the ratios of the ſides. and by this they expreſſed the Propoſition more briefly thus, If there be two equiangular parallelograms, they have to one another the ratio which is the ſame with that which is compounded of ratios that are the ſame with the ratios of the ſides. which is ſhorter than the preceding Enuntiation, but has preciſely the ſame meaning. or yet ſhorter thus; equiangular parallelograms have to one another the ratio which is the ſame with that which is compounded of the ratios of their ſides. and theſe two Enuntiations, the firſt eſpecially, agree to the Demonſtration which is now in the Greek. the Propoſition may be more briefly demonſtrated, as Candalla does, thus; Let ABCD, CEFG be two equiangular parallelograms, and complete the parallelogram CDHG; then, becauſe there are three parallelograms AC, CH, CF, the firſt AC (by the Definition of Compound ratio) has to the third CF, the ratio which is compounded of the ratio of the firſt AC to the ſecond CH, and of the ratio of CH to the third CF; but the parallelogram AC is to the parallelogram CH, as the ſtraight line BC to CG; and the parallelogram CH is to CF, as the ſtraight line CD is to CE; therefore the parallelogram AC has to CF the ratio which is compounded of ratios that are the ſame with the ratios of the ſides. and to this Demonſtration agrees the Enuntiation which is at preſent in the text, viz. equiangular parallelograms have to one another the ratio which is compounded of the ratios of the ſides. for the vulgar reading "which is compounded of their ſides" is abſurd. But in this Edition we have kept the Demonſtration which is in the Greek text, tho' not ſo ſhort as Candalla's; becauſe the way of finding the ratio which



is compounded of the ratios of the ſides; that is, of finding the ratio of the parallelograms, is ſhewn in that, but not in Candalla's Demonſtration; whereby beginners may learn, in like caſes, how to find the ratio which is compounded of two or more given ratios.

From what has been ſaid it may be obſerved, that in any magnitudes whatever of the ſame kind A, B, C, D, &c. the ratio compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, and ſo on to the laſt, is only a name or expreſſion by which the ratio which the firſt A has to the laſt D is ſignified, and by which at the ſame time the ratios of all the magnitudes A to B, B to C, C to D from the firſt to the laſt, to one another, whether they be the ſame, or be not the ſame, are indicated; as in magnitudes which are continual proportionals A, B, C, D, &c. the Duplicate ratio of the firſt to the ſecond is only a name, or expreſſion by which the ratio of the firſt A to the third C is ſignified, and by which, at the ſame time, is ſhewn that there are two ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of the ſecond B to the third or laſt C, which are the ſame with one another; and the Triplicate ratio of the firſt to the ſecond is a name or expreſſion by which the ratio of the firſt A to the fourth D is ſignified, and by which, at the ſame time, is ſhewn that there are three ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of B to the third C, and of C to the fourth or laſt D, which are all the ſame with one another; and ſo in the caſe of any other Multiplicate ratios. And that this is the right explication of the meaning of theſe ratios is plain from the Definitions of Duplicate and Triplicate ratio in which Euclid makes uſe of the word λέγεται, is ſaid to be, or is called; which word, he no doubt made uſe of alſo in the Definition of Compound ratio which Theon, or ſome other, has expunged from the Elements; for the very ſame word is ſtill retained in the wrong Definition of Compound ratio which is now the 5. of the 6. Book. but in the citation of theſe Definitions it is ſometimes retained, as in the Demonſtration of Prop. 19. B. 6. "the firſt is ſaid to have, ἔχειν λέγεται, "to the third the Duplicate ratio," &c. which is wrong tranſlated by Commandine and others "has" inſtead of "is ſaid to have;" and ſometimes it is left out, as in the Demonſtration of Prop. 33. of the 11. Book, in which we find "the firſt has, ἔχει, to "the third the Triplicate ratio;" but without doubt ἔχει, "has," in this place ſignifies the ſame as ἔχειν λέγεται, is ſaid to have. ſo likewiſe in Prop. 23. B. 6. we find this citation "but the ratio of "K to M is compounded, σύγκειται, of the ratio of K to L, and the "ratio of L to M," which is a ſhorter way of expreſſing the ſame thing, which, according to the Definition, ought to have been expreſſed by συγκεῖσθαι λέγεται, is ſaid to be compounded.

From theſe Remarks, together with the Propoſitions ſubjoined to the 5. Book, all that is found concerning Compound ratio either in the antient or modern Geometers may be underſtood and explained.

PROP. XXIV. B. VI.

It ſeems that ſome unſkilful Editor has made up this Demonſtration as we now have it, out of two others; one of which may be made from the 2. Prop. and the other from the 4. of this Book. for after he has from the 2. of this Book, and Compoſition and Permutation, demonſtrated that the ſides about the angle common to the two parallelograms are proportionals, he might have immediately concluded that the ſides about the other equal angles were proportionals, viz. from Prop. 34. B. 1. and Prop. 7. B. 5. this he does not, but proceeds to ſhew that the triangles and parallelograms are equiangular, and in a tedious way, by help of Prop. 4. of this Book, and the 22. of B. 5. deduces the ſame concluſion, from which it is plain that this ill compoſed Demonſtration is not Euclid's. theſe ſuperfluous things are now left out, and a more ſimple Demonſtration is given from the 4. Prop. of this Book, the ſame which is in the Tranſlation from the Arabic, by help of the 2. Prop. and Compoſition; but in this the Author neglects Permutation, and does not ſhew the parallelograms to be equiangular, as is proper to do for the ſake of beginners.

PROP. XXV. B. VI.

It is very evident that the Demonſtration which Euclid had given of this Propoſition, has been vitiated by ſome unſkilful hand. for after this Editor had demonſtrated that "as the rectilineal figure "ABC is to the rectilineal KGH, ſo is the parallelogram BE to the "parallelogram EF," nothing more ſhould have been added but this, "and the rectilineal figure ABC is equal to the parallelogram "BE, therefore the rectilineal KGH is equal to the parallelogram "EF," viz. from Prop. 14. B. 5. but betwixt theſe two ſentences he has inſerted this, "wherefore, by Permutation, as the rectilineal

" figure ABC to the parallelogram BE, ſo is the rectilineal KGH " to the parallelogram EF;" by which, it is plain, he thought it was not ſo evident to conclude that the ſecond of four proportionals is equal to the fourth from the equality of the firſt and third, which is a thing demonſtrated in the 14. Prop. of B. 5. as to conclude that the third is equal to the fourth, from the equality of the firſt and ſecond, which is no where demonſtrated in the Elements as we now have them. but tho' this Propoſition, viz. the third of four proportionals is equal to the fourth, if the firſt be equal to the ſecond, had been given in the Elements by Euclid, as very probably it was, yet he would not have made uſe of it in this place, becauſe, as was ſaid, the concluſion could have been immediately deduced without this ſuperfluous ſtep by Permutation. this we have ſhewn at the greater length, both becauſe it affords a certain proof of the vitiation of the Text of Euclid, for the very ſame blunder is found twice in the Greek Text of Prop. 23. B. 11. and twice in Prop. 2. B. 12. and in the 5. 11. 12. and 18. of that Book; in which places of B. 12. except the laſt of them, it is rightly left out in the Oxford Edition of Commandine's Tranſlation. and alſo that Geometers may beware of making uſe of Permutation in the like caſes, for the Moderns not unfrequently commit this miſtake, and among others Commandine himſelf in his Commentary on Prop. 5. B. 3. p. 6. b. of Pappus Alexandrinus, and in other places. the vulgar notion of proportionals, has, it ſeems, pre-occupied many ſo much, that they do not ſufficiently underſtand the true nature of them.

Beſides, tho' the rectilineal figure ABC, to which another is to be made ſimilar, may be of any kind whatever, yet in the Demonſtration the Greek Text has " triangle" inſtead of " rectilineal " figure," which error is corrected in the above-named Oxford Edition.

PROP. XXVII. B. VI.

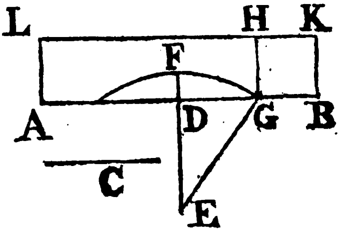
The ſecond Caſe of this has ἄλλως, otherwiſe, prefixed to it, as if it was a different Demonſtration, which probably has been done by ſome unſkilful Librarian. Dr. Gregory has rightly left it out. the ſcheme of this ſecond Caſe ought to be marked with the ſame letters of the Alphabet which are in the ſcheme of the firſt, as is now done.

PROP. XXVIII. and XXIX. B. VI.

Theſe two Problems, to the firſt of which the 27. Prop. is neceſſary, are the moſt general and uſeful of all in the Elements, and are moſt frequently made uſe of by the antient Geometers in the ſolution of other Problems; and therefore are very ignorantly left out by Tacquet and Dechales in their Editions of the Elements; who pretend that they are ſcarce of any uſe. the Caſes of theſe Problems, wherein it is required to apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, either deficient or exceeding by a ſquare; as alſo to apply a rectangle which ſhall be equal to another given, to a given ſtraight line, deficient or exceeding by a ſquare, are very often made uſe of by Geometers. and on this account, it is thought proper, for the ſake of beginners, to give their conſtructions, as follows.

1. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, deficient by a ſquare. but the given ſquare muſt not be greater than that upon the half of the given line.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal, and this ſquare by the determination, is not greater than that upon half of the ſtraight line AB.



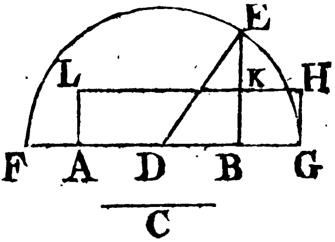
Biſect AB in D, and if the ſquare upon AD be equal to the ſquare upon C, the thing required is done. but if it be not equal to it, AD muſt be greater than C, according to the determination. draw DE at right angles to AB, and make it equal to C; produce ED to F, ſo that EF be equal to AD or DB, and from the center E, at the diſtance EF deſcribe a circle meeting AB in G, and upon GB deſcribe the ſquare GBKH, and complete the rectangle AGHL; alſo join EG. and becauſe AB is biſected in D, the rectangle AG, GB together with the ſquare of DG is equal [a] to (the ſquare of DB, that is of EF or EG, that is a. 5. 2.
to) the ſquares of ED, DG. take away the ſquare of DG from each of theſe equals, therefore the remaining rectangle AG, GB is equal to the ſquare of ED, that is, of C. but the rectangle AG, GB is the rectangle AH, becauſe GH is equal to GB. therefore the rec-

tangle AH is equal to the given ſquare upon the ſtraight line C, wherefore the rectangle AH equal to the given ſquare upon C, has been applied to the given ſtraight line AB, deficient by the ſquare GK. Which was to be done.

2. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, exceeding by a ſquare.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal.

Biſect AB in D, and draw BE at right angles to it, ſo that BE be equal to C, and, having joined DE, from the center D at the diſtance DE deſcribe a circle meeting AB produced in G; upon BG deſcribe the ſquare BGHK, and complete the rectangle AGHL. and becauſe AB is biſected in D, and produced to G, the rectangle AG, GB together with the ſquare of DB is equal [a] to (the ſquare of DG, or DE, that is to) the ſquares of EB, BD. from each of theſe equals take the ſquare of DB, therefore the remaining rectangle AG, GB is equal to the ſquare of BE, that is to the ſquare upon C. but the rectangle AG, GB is the rectangle AH, becauſe GH is equal to GB. therefore the rectangle AH is equal to the ſquare upon C. wherefore the rectangle AH equal to the given ſquare upon C, has been applied to the given ſtraight line AB, exceeding by the ſquare GK. Which was to be done.



[a] 6. 2.

3. To apply a rectangle to a given ſtraight line which ſhall be equal to a given rectangle, and be deficient by a ſquare. but the given rectangle muſt not be greater than the ſquare upon the half of the given ſtraight line.

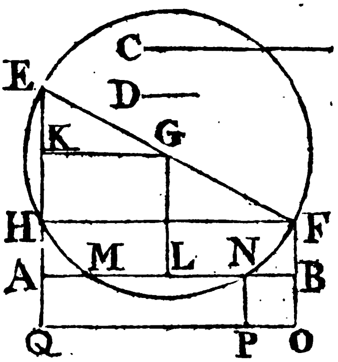
Let AB be the given ſtraight line, and let the given rectangle be that which is contained by the ſtraight lines C, D, which is not greater than the ſquare upon the half of AB. it is required to apply to AB a rectangle equal to the rectangle C, D, deficient by a ſquare.

Draw AE, BF at right angles to AB, upon the ſame ſide of it, and make AE equal to C, and BF to D. join EF and biſect it in G, and from the center G, at the diſtance GE deſcribe a circle meeting

AE again in H; join HF and draw GK parallel to it, and GL parallel to AE meeting AB in L.

Becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and AH and BF are parallels, wherefore AH is equal to BF, and the rectangle EA, AH equal to the rectangle EA, BF, that is to the rectangle C, D. and becauſe EG, GF are equal to one another, and AE, LG, BF parallels, therefore AL and LB are equal; alſo EK is equal to KH [a], and the rectangle C, D, from the determination, is not greater than the ſquare of AL the half of AB, wherefore the rectangle EA, AH is not greater than the ſquare of AL, that is of KG. add to each the ſquare of KE, therefore the ſquare [b] of AK is not greater than the ſquares of EK, KG, that is than the ſquare of EG; and conſequently the ſtraight line AK or GL is not greater than GE. Now, if GE be equal to GL, the circle EHF touches AB in L, and therefore the ſquare of AL is [c] equal to the rectangle EA, AH, that is to the given rectangle C, D; and that which was required is done. but if EG, GL be unequal, EG muſt be the greater, and therefore the circle EHF cuts the ſtraight line AB; let it cut it in the points M, N, and upon NB deſcribe the ſquare NBOP, and complete the rectangle ANPQ. becauſe ML is equal to [d] LN, and it has been proved that AL is equal to LB, therefore AM is equal to NB, and the rectangle AN, NB equal to the rectangle NA, AM, that is to the rectangle [e] EA, AH or the rectangle C, D. but the rectangle AN, NB is the rectangle AP, becauſe PN is equal to NB. therefore the rectangle AP is equal to the rectangle C, D, and the rectangle AP equal to the given rectangle C, D has been applied to the given ſtraight line AB, deficient by the ſquare BP. Which was to be done.

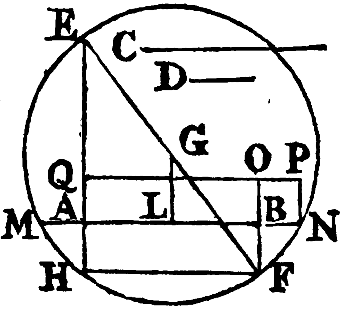
a. 3. 3.
b. 6. 2.
c. 36. 3.
d. 3. 3.
e. Cor. 36. 3.



4. To apply a rectangle to a given ſtraight line that ſhall be equal to a given rectangle, exceeding by a ſquare.

Let AB be the given ſtraight line, and the rectangle C, D the given rectangle, it is required to apply a rectangle to AB equal to C, D, exceeding by a ſquare.

Draw AE, BF at right angles to AB, on the contrary ſides of it, and make AE equal to C, and BF equal to D. join EF and biſect it in G, and from the center G, at the diſtance GE deſcribe a circle meeting AE again in H; join HF, and draw GL parallel to AE; let the circle meet AB produced in M, N, and upon BN deſcribe the ſquare NBOP, and complete the rectangle ANPQ. becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and therefore AH and BF are equal, and the rectangle EA, AH equal to the rectangle EA, BF, that is to the rectangle C, D. and becauſe ML is equal to LN, and AL to LB, therefore MA is equal to BN, and the rectangle AN, NB to MA, AN,
a 35. 3. that is [a] to the rectangle EA, AH or the rectangle C, D. therefore the rectangle AN, NB, that is AP is equal to the rectangle C, D; and to the given ſtraight line AB the rectangle AP has been applied equal to the given rectangle C, D, exceeding by the ſquare BP. Which was to be done.



Willebrordus Snellius was the firſt, as far as I know, who gave theſe conſtructions of the 3. and 4. Problems in his Apollonius Batavus. and afterwards the learned Dr. Halley gave them in the Scholium of the 18. Prop. of the 8. B. of Apollonius's Conics reſtored by him.

The 3. Problem is otherwiſe enuntiated thus, To cut a given ſtraight line AB in the point N, ſo as to make the rectangle AN, NB equal to a given ſpace. or, which is the ſame thing, Having given AB the ſum of the ſides of a rectangle, and the magnitude of it being likewiſe given, to find its ſides.

And the 4. Problem is the ſame with this, To find a point N in the given ſtraight line AB produced, ſo as to make the rectangle AN, NB equal to a given ſpace. or, which is the ſame thing, Having given AB the difference of the ſides of a rectangle, and the magnitude of it, to find the ſides.

PROP. XXXI. B. VI.

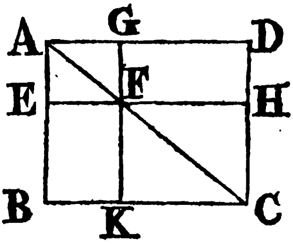
In the Demonſtration of this the inverſion of proportionals is twice neglected, and is now added, that the concluſion may be legitimately made by help of the 24. Prop. of B. 5. as Clavius had done.

PROP. XXXII. B. VI.

The Enuntiation of the preceding 26. Prop. is not general enough; becauſe not only two ſimilar parallelograms that have an angle common to both, are about the ſame diameter; but likewiſe two ſimilar parallelograms that have vertically oppoſite angles, have their diameters in the ſame ſtraight line. but there ſeems to have been another, and that a direct Demonſtration of theſe caſes, to which this 32. Propoſition was needful. and the 32. may be otherwiſe and ſomething more briefly demonſtrated as follows.

PROP. XXXII. B. VI.

If two triangles which have two ſides of the one, &c.



Let GAF, HFC be two triangles which have two ſides AG, GF proportional to the two ſides FH, HC, viz. AG to GF, as FH to HC; and let AG be parallel to FH, and GF to HC; AF and FC are in a ſtraight line.

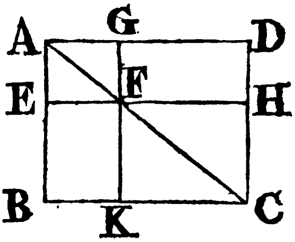
Draw CK parallel [a] to FH, and let a. 31. 1.
it meet GF produced in K. becauſe AG,
KC are each of them parallel to FH,
they are parallel [b] to one another, and b. 30. 1.
therefore the alternate angles AGF,
FKC are equal. and AG is to GF, as (FH to HC, that is [c]) CK c. 34. 1.
to KF; wherefore the triangles AGF, CKF are equiangular [d], and d. 6. 6.
the angle AFG equal to the angle CFK. but GFK is a ſtraight
line, therefore AF and FC are in a ſtraight line [e]. e. 14. 1.

The 26. Prop. is demonſtrated from the 32. as follows.

If two ſimilar and ſimilarly placed parallelograms have an angle common to both, or vertically oppoſite angles; their diameters are in the ſame ſtraight line.

Firſt, Let the parallelograms ABCD, AEFG have the angle BAD common to both, and be ſimilar, and ſimilarly placed; ABCD, AEFG are about the ſame diameter.

Produce EF, GF, to H, K, and join FA, FC. then becauſe the parallelograms ABCD, AEFG are ſimilar, DA is to AB, as GA to AE; wherefore the remainder DG is [a] to the remainder EB, as GA to AE. but DG is equal to FH, EB to HC, and AE to GF. therefore as FH to HC, ſo is AG to GF; and FH, HC are parallel to AG, GF; and the triangles AGF, FHC are joined at one angle, in the point F; wherefore AF, FC are in the ſame ſtraight line [b].



a. Cor. 19. 5.

b. 32. 6.

Next, Let the parallelograms KFHC, GFEA which are ſimilar and ſimilarly placed, have their angles KFH, GFE vertically oppoſite; their diameters AF, FC are in the ſame ſtraight line.

Becauſe AG, GF are parallel to FH, HC; and that AG is to GF, as FH to HC; therefore AF, FC are in the ſame ſtraight line [b].

PROP. XXXIII. B. VI.

The words "becauſe they are at the center," are left out, as the addition of ſome unſkilful hand.

In the Greek, as alſo in the Latin Tranſlation, the words ὡς ἔτυχε, "any whatever," are left out in the Demonſtration of both parts of the Propoſition, and are now added as quite neceſſary. and in the Demonſtration of the ſecond part, where the triangle BGC is proved to be equal to CGK, the illative particle ἄρα in the Greek Text ought to be omitted.

The ſecond part of the Propoſition is an addition of Theon's, as he tell us in his Commentary on Ptolomy's Μεγάλῃ Συντάξει, p. 50.

PROP. B, C, D. B. VI.

Theſe three Propoſitions are added, becauſe they are frequently made uſe of by Geometers.

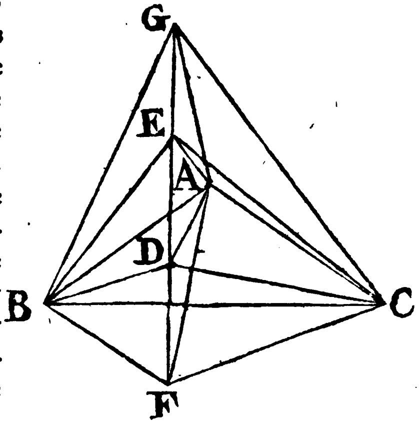
DEF. IX. and XI. B. XI.

THE ſimilitude of plane figures is defined from the equality of their angles, and the proportionality of the ſides about the equal angles; for from the proportionality of the ſides only, or only from the equality of the angles, the ſimilitude of the figures does not follow, except in the caſe when the figures are triangles. the ſimilar poſition of the ſides, which contain the figures, to one another, depending partly upon each of theſe. and, for the ſame reaſon, thoſe are ſimilar ſolid figures which have all their ſolid angles equal, each to each, and are contained by the ſame number of ſimilar plane figures. for there are ſome ſolid figures contained by ſimilar plane figures, of the ſame number, and even of the ſame magnitude, that are neither ſimilar nor equal, as ſhall be demonſtrated after the Notes on the 10. Definition. upon this account it was neceſſary to amend the Definition of ſimilar ſolid figures, and to place the Definition of a ſolid angle before it. and from this and the 10. Definition, it is ſufficiently plain how much the Elements have been ſpoiled by unſkilful Editors.

DEF. X. B. XI.

Since the meaning of the word "equal" is known and eſtabliſhed before it comes to be uſed in this Definition, therefore the Propoſition which is the 10. Definition of this Book, is a Theorem the truth or falſhood of which ought to be demonſtrated, not aſſumed; ſo that Theon, or ſome other Editor, has ignorantly turned a Theorem which ought to be demonſtrated into this 10. Definition. that figures are ſimilar, ought to be proved from the Definition of ſimilar figures; that they are equal ought to be demonſtrated from the Axiom, "Magnitudes that wholly coincide, are equal to one another;" or from Prop. A. of Book 5. or the 9. Prop. or the 14. of the ſame Book, from one of which the equality of all kind of figures muſt ultimately be deduced. In the preceding Books, Euclid has given no Definition of equal figures, and it is certain he did not give this. for what is called the 1. Def. of the 3. Book, is really a Theorem in which theſe circles are ſaid to be equal, that have the ſtraight lines from their centers to the circumferences equal, which is plain from the Definition of a circle, and therefore has by ſome Editor been improperly placed among the Definitions. The equality of figures ought not to be defined, but demonſtrated. therefore tho' it were true that ſolid figures contained by the ſame number of ſimilar and equal plane figures are equal to one another, yet he would juſtly deſerve to be blamed who ſhould make a Definition of this Propoſition which ought to be demonſtrated. But if this Propoſition be not true, muſt it not be confeſſed that Geometers have for theſe thirteen hundred years been miſtaken in this Elementary matter? and this ſhould teach us modeſty, and to acknowledge how little, thro' the weakneſs of our minds, we are able to prevent miſtakes even in the principles of ſciences which are juſtly reckoned amongſt the moſt certain; for that the Propoſition is not univerſally true can be ſhewn by many examples; the following is ſufficient.

Let there be any plane rectilineal figure, as the triangle ABC, a. 12. 11. and from a point D within it draw [a] the ſtraight line DE at right angles to the plane ABC; in DE take DE, DF equal to one another, upon the oppoſite ſides of the plane, and let G be any point in EF; join DA, DB, DC; EA, EB, EC; FA, FB, FC; GA, GB, GC. becauſe the ſtraight line EDF is at right angles to the plane ABC, it makes right angles with DA, DB, DC which it meets in that plane; and in the triangles EDB, FDB, ED and DB are equal to FD and DB, each to each, and they contain right angles, there- b. 4. 1. fore the baſe EB is equal [b] to the baſe FB; in the ſame manner EA is equal to FA, and EC to FC. and in the triangles EBA, FBA, EB, BA are equal to FB, BA, and the baſe EA is equal to the baſe FA; wherefore the c. 8. 1. angle EBA is equal [c] to the angle FBA, and the triangle EBA equal [b] to the triangle FBA, and the other angles equal to the other angles; therefore theſe tri- d. {4. 6. 1. Def. 6. angles are ſimilar [d]. in the ſame manner the triangle EBC is ſimilar to the triangle FBC, and the triangle EAC to FAC. therefore there are two ſolid figures each of which is contained by ſix triangles, one of them by three triangles the common vertex of which is the point G, and their baſes the ſtraight lines AB, BC, CA; and by three other triangles the common vertex of which is the point E, and their baſes the ſame lines AB, BC, CA. the other ſolid is contained by the ſame three triangles the common vertex of which is G, and their baſes AB, BC, CA; and by three other triangles of which the common vertex is the point F, and their



baſes the ſame ſtraight lines AB, BC, CA. now the three triangles GAB, GBC, GCA are common to both ſolids, and the three others EAB, EBC, ECA of the firſt ſolid have been ſhewn equal and ſimilar to the three others FAB, FBC, FCA of the other ſolid, each to each. therefore theſe two ſolids are contained by the ſame number of equal and ſimilar planes. but that they are not equal is manifeſt, becauſe the firſt of them is contained in the other. therefore it is not univerſally true that ſolids are equal which are contained by the ſame number of equal and ſimilar planes.

Cor. From this it appears that two unequal ſolid angles may be contained by the ſame number of equal plane angles.

For the ſolid angle at B which is contained by the four plane angles EBA, EBC, GBA, GBC is not equal to the ſolid angle at the ſame point B which is contained by the four plane angles FBA, FBC, GBA, GBC; for this laſt contains the other. and each of them is contained by four plane angles, which are equal to one another, each to each, or are the ſelf ſame; as has been proved. and, indeed, there may be innumerable ſolid angles all unequal to one another, which are each of them contained by plane angles that are equal to one another, each to each. it is likewiſe manifeſt that the before-mentioned ſolids are not ſimilar, ſince their ſolid angles are not all equal.

And that there may be innumerable ſolid angles all unequal to one another, which are each of them contained by the ſame plane angles diſpoſed in the ſame order, will be plain from the three following Propoſitions.

PROP. I. PROBLEM.

Three magnitudes A, B, C being given, to find a fourth ſuch, that every three ſhall be greater than the remaining one.

Let D be the fourth, therefore D muſt be leſs than A, B, C together. of the three A, B, C let A be that which is not leſs than either of the two B and C. and firſt, let B and C together be not leſs than A; therefore B, C, D together are greater than A. and becauſe A is not leſs than B; A, C, D together are greater than B. in the like manner A, B, D together are greater than C. wherefore in the caſe in which B and C together are not leſs than A, any magnitude D which is leſs than A, B, C together will anſwer the Problem.

But if B and C together be leſs than A, then becauſe it is required that B, C, D together be greater than A, from each of theſe taking away B, C, the remaining one D muſt be greater than the exceſs of A above B and C. take therefore any magnitude D which is leſs than A, B, C together, but greater than the exceſs of A above B and C. then B, C, D together are greater than A; and becauſe A is greater than either B or C, much more will A and D, together with either of the two B, C be greater than the other. and, by the conſtruction, A, B, C are together greater than D.

Cor. If beſides, it be required that A and B together ſhall not be leſs than C and D together; the exceſs of A and B together above C muſt not be leſs than D, that is D muſt not be greater than that exceſs.

PROP. II. PROBLEM.

Four magnitudes A, B, C, D being given, of which A and B together are not leſs than C and D together, and ſuch that any three of them whatever are greater than the fourth; it is required to find a fifth magnitude E ſuch, that any two of the three A, B, E ſhall be greater than the third, and alſo that any two of the three C, D, E ſhall be greater than the third. Let A be not leſs than B, and C not leſs than D.

Firſt, Let the exceſs of C above D be not leſs than the exceſs of A above B. it is plain that a magnitude E can be taken which is leſs than the ſum of C and D, but greater than the exceſs of C above D; let it be taken, then E is greater likewiſe than the exceſs of A above B; wherefore E and B together are greater than A; and A is not leſs than B, therefore A and E together are greater than B. and, by the Hypotheſis, A and B together are not leſs than C and D together, and C and D together are greater than E; therefore likewiſe A and B are greater than E.

But let the exceſs of A above B be greater than the exceſs of C above D. and, becauſe, by the Hypotheſis, the three B, C, D are together greater than the fourth A; C and D together are greater than the exceſs of A above B. therefore a magnitude may be taken which is leſs than C and D together, but greater than the exceſs of A above B. Let this magnitude be E, and becauſe E is greater than the exceſs of A above B, B together with E is greater than A. and, as in the preceding caſe, it may be ſhewn that A together with E is greater than B, and that A together with B is greater than E.

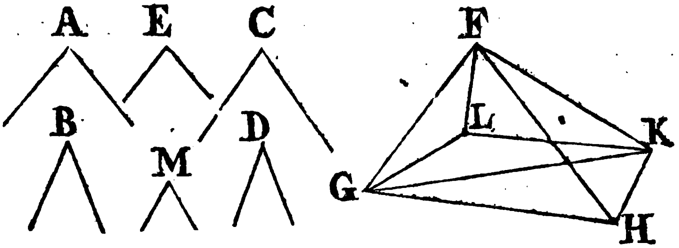
therefore in each of the caſes it has been ſhewn that any two of the three A, B, E are greater than the third.

And becauſe in each of the caſes E is greater than the exceſs of C above D, E together with D is greater than C, and, by the Hypotheſis, C is not leſs than D, therefore E together with C is greater than D; and, by the conſtruction, C and D together are greater than E. therefore any two of the three, C, D, E are greater than the third.

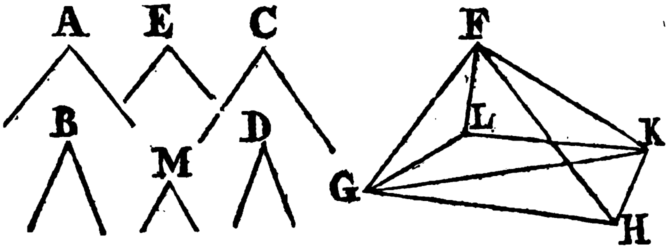
PROP. III. THEOREM.

There may be innumerable ſolid angles all unequal to one another, each of which is contained by the ſame four plane angles, placed in the ſame order.

Take three plane angles A, B, C, of which A is not leſs than either of the other two, and ſuch, that A and B together are leſs than two right angles; and by Problem 1. and its Corollary, find a fourth angle D ſuch, that any three whatever of the angles A, B, C, D be greater than the remaining angle, and ſuch, that A and B together be not leſs than C and D together. and by Problem 2. find a fifth angle E ſuch, that any two of the angles A, B, E be greater than the third, and alſo that any two of the angles C, D, E be greater than the third. and becauſe A and B together are leſs than two right angles, the double of A and B together is leſs than four right angles. but A and B together are greater than the angle E, wherefore the double of A and B together is greater than the three angles A, B, E together, which three are conſequently leſs than four right angles; and every two of the ſame angles A, B, E are greater than the third; therefore, by Prop. 23. 11. a ſolid angle may be made contained by three plane angles equal to the angles A, B, E, each to each. Let this be the angle F contained by the three plane angles GFH, HFK, GFK which are equal to the angles

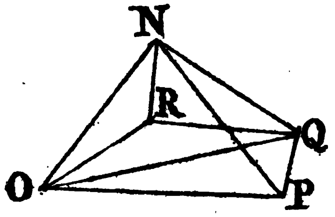


Book XI. A, B, E, each to each. and becauſe the angles C, D together are not greater than the angles A, B together, therefore the angles C, D, E are not greater than the angles A, B, E. but theſe laſt three are leſs than four right angles, as has been demonſtrated, wherefore alſo the angles C, D, E are together leſs than four right angles, and every two of them are greater than the third; therefore a ſolid angle may be made which ſhall be contained by three plane angles a. 23. 11. equal to the angles C, D, E, each to each[a]. and by Prop. 26. 11.



at the point F in the ſtraight line FG a ſolid angle may be made equal to that which is contained by the three plane angles that are equal to the angles C, D, E. let this be made, and let the angle GFK, which is equal to E, be one of the three; and let KFL, GFL be the other two which are equal to the angles C, D, each to each. thus, there is a ſolid angle conſtituted at the point F contained by the four plane angles GFH, HFK, KFL, GFL which are equal to the angles A, B, C, D, each to each.

Again, Find another angle M ſuch, that every two of the three angles A, B, M be greater than the third, and alſo every two of the three C, D, M be greater than the third. and, as in the preceding part, it may be demonſtrated that the three A, B, M, are leſs than four right angles, as alſo that the three C, D, M are leſs than four right angles. Make therefore[a] a ſolid angle at N contained by the three plane angles ONP, PNQ, ONQ, which are equal to A, B, M, each to each. and by Prop. 26. 11. make at the point N in the ſtraight line ON a ſolid angle contained by three plane angles of which one is the angle ONQ equal to M, and the other two are the angles QNR, ONR which are



equal to the angles C, D, each to each. thus at the point N there is a ſolid angle contained by the four plane angles ONP, PNQ, QNR, ONR which are equal to the angles A, B, C, D, each to each. and that the two ſolid angles at the points F, N, each of which is contained by the above named four plane angles, are not equal to one another, or that they cannot coincide, will be plain by conſidering that the angles GFK, ONQ; that is, the angles E, M are unequal by the conſtruction, and therefore the ſtraight lines GF, FK cannot coincide with ON, NQ, nor conſequently can the ſolid angles, which therefore are unequal.

And becauſe from the three given plane angles A, B, C there can be found innumerable other angles that will ſerve the ſame purpoſe with the angle D, and again from D or any one of theſe others, and the angles A, B, C, there may be found innumerable angles, ſuch as E or M; it is plain that innumerable other ſolid angles may be conſtituted which are each contained by the ſame four plane angles, and all of them unequal to another. Q. E D.

And from this it appears that Clavius and other Authors are miſtaken who aſſert that thoſe ſolid angles are equal which are contained by the ſame number of plane angles that are equal to one another, each to each. alſo it is plain that the 26. Prop. of Book 11. is by no means ſufficiently demonſtrated, becauſe the equality of two ſolid angles, whereof each is contained by three plane angles which are equal to one another, each to each, is only aſſumed, and not demonſtrated.

PROP. I. B. XI.

The words at the end of this, "for a ſtraight line cannot meet "a ſtraight line in more than one point," are left out, as an addition by ſome unſkilful hand; for this is to be demonſtrated, not aſſumed.

Mr. Thomas Simpſon, in his notes at the end of the 2d Edition of his Elements of Geometry, p. 262. after repeating the words of this note, adds "Now can it poſſibly ſhew any want of ſkill in an "editor" (he means Euclid or Theon) "to refer to an Axiom which "Euclid himſelf had laid down Book 1. N° 14. (he means Barrow's Euclid, for it is the 10th in the Greek) "and not to have "demonſtrated, what no man can demonſtrate?" But all that in this caſe can follow from that Axiom is, that if two ſtraight lines could meet each other in two points, the parts of them betwixt theſe points muſt coincide, and ſo they would have a ſegment betwixt theſe points common to both. Now, as it has not been ſhewn in Euclid, that they cannot have a common ſegment, this does not prove that they cannot meet in two points, from which their not having a common ſegment is deduced in the Greek Edition. but, on the contrary, becauſe they cannot have a common ſegment, as is ſhewn in Cor. of 11. Prop. B. 1. of 4to. Edition, it follows plainly that they cannot meet in two points, which the remarker ſays no man can demonſtrate.

Mr. Simpſon in the ſame notes, p. 265. juſtly obſerves that in the Corollary of Prop. 11. Book 1. 4to. Edit: the ſtraight lines AB, BD, BC, are ſuppoſed to be all in the ſame plane, which cannot be aſſumed in 1. Prop. B. 11. this, ſoon after the 4to. Edition was publiſhed, I obſerved and corrected as it is now in this Edition. he is miſtaken in thinking the 10th Axiom he mentions here, to be Euclid's; it is none of Euclid's, but is the 10th in Dr. Barrow's Edition, who had it from Herigon's Curſus Vol. 1. and in place of it the Corollary of 11. Prop. Book 1. was added.

PROP. II. B. XI.

This Propoſition ſeems to have been changed and vitiated by ſome Editor; for all the figures defined in the 1. Book of the Elements, and among them triangles, are, by the Hypotheſis, plane figures; that is, ſuch as are deſcribed in a plane; wherefore the ſecond part of the Enuntiation needs no Demonſtration. beſides a convex ſuperficies may be terminated by three ſtraight lines meeting one another. the thing that ſhould have been demonſtrated is, that two, or three ſtraight lines that meet one another, are in one plane. and as this is not ſufficiently done, the Enuntiation and Demonſtration are changed into thoſe now put into the Text.

PROP. III. B. XI.

In this Propoſition the following words near to the end of it are left out, viz. "therefore DEB, DFB are not ſtraight lines, in the "like manner it may be demonſtrated that there can be no other "ſtraight line between the points D, B." becauſe from this that two lines include a ſpace, it only follows that one of them is not a ſtraight line. and the force of the argument lies in this, viz. if the common ſection of the planes be not a ſtraight line, then two ſtraight lines could include a ſpace, which is abſurd; therefore the common ſection is a ſtraight line.

PROP. IV. B. XI.

The words "and the triangle AED to the triangle BEC" are omitted, becauſe the whole concluſion of the 4. Prop. B. 1. has been ſo often repeated in the preceding Books, it was needleſs to repeat it here.

PROP. V. B. XI.

In this, near to the end, ἐπιπέδῳ, ought to be left out in the Greek text. and the word "plane" is rightly left out in the Oxford Edition of Commandine's Tranſlation.

PROP. VII. B. XI.

This Propoſition has been put into this Book by ſome unſkilful Editor, as is evident from this, that ſtraight lines which are drawn from one point to another in a plane, are, in the preceding Books, ſuppoſed to be in that plane. and if they were not, ſome Demonſtrations in which one ſtraight line is ſuppoſed to meet another would not be concluſive, becauſe theſe lines would not meet one another. for inſtance, in Prop. 30. Book 1. the ſtraight line GK would not meet EF, if GK were not in the plane in which are the parallels AB, CD, and in which, by Hypotheſis, the ſtraight line EF is. beſides, this 7. Propoſition is demonſtrated by the preceding 3. in which the very thing which is propoſed to be demonſtrated in the 7. is twice aſſumed, viz. that the ſtraight line drawn from one point to another in a plane, is in that plane; and the ſame thing is aſſumed in the preceding 6. Prop. in which the ſtraight line which joins the points B, D that are in the plane to which AB and CD are at right angles, is ſuppoſed to be in that plane. and the 7. of which another Demonſtration is given, is kept in the Book merely to preſerve the number of the Propoſitions; for it is evident from the 7. and 35. Definitions of the 1. Book, tho' it had not been in the Elements.

PROP. VIII. B. XI.

In the Greek, and in Commandine's and Dr. Gregory's Tranſlations, near to the end of this Propoſition, are the following words, "but DC is in the plane thro' BA, AD" inſtead of which in the Oxford edition of Commandine's tranſlation, is rightly put "but "DC is in the plane thro' BD, DA." but all the Editions have the following words, viz. "becauſe AB, BD are in the plane thro' "BD, DA, and DC is in the plane in which are AB, BD," which are manifeſtly corrupted, or have been added to the Text; for there was not the leaſt neceſſity to go ſo far about to ſhew that DC is in the ſame plane in which are BD, DA, becauſe it immediately follows from Prop. 7. preceding, that BD, DA are in the plane in which are the parallels AB, CD. therefore inſtead of theſe words there ought only to be "becauſe all three are in the plane in "which are the parallels AB, CD."

PROP. XV. B. XI.

After the words, "and becauſe BA is parallel to GH," the following are added "for each of them is parallel to DE, and are "not both in the ſame plane with it," as being manifeſtly forgotten to be put into the Text.

PROP. XVI. B. XI.

In this, near to the end, inſtead of the words "but ſtraight lines "which meet neither way" ought to be read "but ſtraight lines "in the ſame plane which produced meet neither way." becauſe tho' in citing this Definition in Prop. 27. B. 1. it was not neceſſary to mention the words, "in the ſame plane" all the ſtraight lines in the Books preceding this being in the ſame plane; yet here it was quite neceſſary.

PROP. XX. B. XI.

In this, near the beginning, are the words, "but if not, let BAC "be the greater." but the angle BAC may happen to be equal to one of the other two. wherefore this place ſhould be read thus, "but if not, let the angle BAC be not leſs than either of the other "two, but greater than DAB."

At the end of this Propoſition it is ſaid, "in the ſame manner "it may be demonſtrated," tho' there is no need of any Demonſtration; becauſe the angle BAC being not leſs than either of the other two, it is evident that BAC together with one of them is greater than the other.

PROP. XXII. B. XI.

And likewiſe in this, near the beginning, it is ſaid, "but if not, "let the angles at B, E, H be unequal, and let the angle at B be "greater than either of thoſe at E, H." which words manifeſtly ſhew this place to be vitiated, becauſe the angle at B may be equal to one of the other two. they ought therefore to be read thus, "but "if not, let the angles at B, E, H be unequal, and let the angle "at B be not leſs than either of the other two at E, H. therefore "the ſtraight line AC is not leſs than either of the two DF, GK."

PROP. XXIII. B. XI.

The Demonſtration of this is made ſomething ſhorter, by not repeating in the third caſe the things which were demonſtrated in the firſt; and by making uſe of the conſtruction which Campanus has given; but he does not demonſtrate the ſecond and third caſes. the conſtruction and demonſtration of the third caſe are made a little more ſimple than in the Greek Text.

PROP. XXIV. B. XI.

The word "ſimilar" is added to the Enuntiation of this Propoſition, becauſe the planes containing the ſolids which are to be demonſtrated to be equal to one another in the 25. Propoſition, ought to be ſimilar and equal; that the equality of the ſolids may be inferred from Prop. C. of this Book. and in the Oxford Edition of Commandine's Tranſlation, a Corollary is added to Prop. 24. to ſhew that the parallelograms mentioned in this Propoſition are ſimilar, that the equality of the ſolids in Prop. 25. may be deduced from the 10. Def. of B. 11.

PROP. XXV. and XXVI. B. XI.

In the 25. Prop. ſolid figures which are contained by the ſame number of ſimilar and equal plane figures, are ſuppoſed to be equal to one another. and it ſeems that Theon, or ſome other Editor, that he might ſave himſelf the trouble of demonſtrating the ſolid figures mentioned in this Propoſition to be equal to one another, has inſerted the 10. Def. of this Book, to ſerve inſtead of a Demonſtration; which was very ignorantly done.

Likewiſe in the 26. Prop. two ſolid angles are ſuppoſed to be equal, if each of them be contained by three plane angles which are equal to one another, each to each. and it is ſtrange enough, that none of the Commentators on Euclid have, as far as I know, perceived that ſomething is wanting in the demonſtrations of theſe two Propoſitions. Clavius, indeed, in a note upon the 11. Def.

of this Book, affirms, that it is evident that thoſe ſolid angles are equal which are contained by the ſame number of plane angles, equal to one another, each to each, becauſe they will coincide, if they be conceived to be placed within one another; but this is ſaid without any proof, nor is it always true, except when the ſolid angles are contained by three plane angles only, which are equal to one another, each to each. and in this caſe the propoſition is the ſame with this, that two ſpherical triangles that are equilateral to one another, are alſo equiangular to one another, and can coincide; which ought not to be granted without a demonſtration. Euclid does not aſſume this in the caſe of rectilineal triangles, but demonſtrates in Prop. 8. B. 1. that triangles which are equilateral to one another are alſo equiangular to one another; and from this their total equality appears by Prop. 4. B. 1. and Menelaus, in the 4. Prop. of his 1. Book of Spherics, explicitly demonſtrates that ſpherical triangles which are mutually equilateral, are alſo equiangular to one another; from which it is eaſy to ſhew that they muſt coincide, providing they have their ſides diſpoſed in the ſame order and ſituation.

To ſupply theſe defects, it was neceſſary to add the three Propoſitions marked A, B, C to this Book. for the 25. 26. and 28. Propoſitions of it, and conſequently eight others, viz. the 27. 31. 32. 33. 34. 36. 37. and 40. of the ſame, which depend upon them, have hitherto ſtood upon an infirm foundation; as alſo, the 8. 12. Cor. of 17. and 18. of the 12. Book, which depend upon the 9. Definition. for it has been ſhewn in the Notes on Def. 10. of this Book, that ſolid figures which are contained by the ſame number of ſimilar and equal plane figures, as alſo ſolid angles that are contained by the ſame number of equal plane angles are not always equal to one another.

It is to be obſerved that Tacquet, in his Euclid, defines equal ſolid angles to be ſuch, "as being put within one another do coincide." but this is an Axiom, not a Definition, for it is true of all magnitudes whatever. he made this uſeleſs Definition, that by it he might demonſtrate the 36. Prop. of this Book without the help of the 35. of the ſame. concerning which Demonſtration, ſee the Note upon Prop. 36.

PROP. XXVIII. B. XI.

In this it ought to have been demonſtrated, not aſſumed, that the diagonals are in one plane. Clavius has ſupplied this defect,

PROP. XXIX. B. XI.

There are three Caſes of this Propoſition; the firſt is when the two parallelograms oppoſite to the baſe AB have a ſide common to both; the ſecond is, when theſe parallelograms are ſeparated from one another; and the third, when there is a part of them common to both; and to this laſt only the Demonſtration that has hitherto been in the Elements does agree. The firſt Caſe is immediately deduced from the preceding 28. Propoſition, which ſeems for this purpoſe to have been premiſed to this 29. for it is neceſſary to none but to it, and to the 40. of this Book, as we now have it, to which laſt it would, without doubt, have been premiſed, if Euclid had not made uſe of it in the 29. but ſome unſkilful Editor has taken it away from the Elements, and has mutilated Euclid's Demonſtration of the other two Caſes, which is now reſtored, and ſerves for both at once.

PROP. XXX. B. XI.

In the Demonſtration of this, the oppoſite planes of the ſolid CP, in the figure in this Edition; that is, of the ſolid CO in Commandine's figure, are not proved to be parallel; which it is proper to do for the ſake of learners.

PROP. XXXI. B. XI.

There are two Caſes of this Propoſition; the firſt is when the inſiſting ſtraight lines are at right angles to the baſes; the other when they are not. the firſt Caſe is divided again into two others, one of which is when the baſes are equiangular parallelograms; the other when they are not equiangular. the Greek Editor makes no mention of the firſt of theſe two laſt Caſes, but has inſerted the Demonſtration of it as a part of that of the other. and therefore ſhould have taken notice of it in a Corollary; but we thought it better to give theſe two caſes ſeparately. the Demonſtration alſo is made ſomething ſhorter by following the way Euclid has made uſe of in Prop. 14. B. 6. beſides, in the Demonſtration of the caſe in which the inſiſting ſtraight lines are not at right angles to the baſes, the Editor does not prove that the ſolids deſcribed in the conſtruction are parallelepipeds, which it is not to be thought that Euclid neglected. alſo the words, " of which the inſiſting ſtraight lines are "not in the ſame ſtraight lines," have been added by ſome unſkilful hand; for they may be in the ſame ſtraight lines.

PROP. XXXII. B. XI.

The Editor has forgot to order the parallelogram FH to be applied in the angle FGH equal to the angle LCG, which is neceſſary. Clavius has ſupplied this.

Alſo, in the conſtruction, it is required to complete the ſolid of which the baſe is FH, and altitude the ſame with that of the ſolid CD; but this does not determine the ſolid to be completed, ſince there may be innumerable ſolids upon the ſame baſe, and of the ſame altitude. it ought therefore to be ſaid "complete the ſolid of "which the baſe is FH, and one of its inſiſting ſtraight lines is "FD." the ſame correction muſt be made in the following Propoſition 33.

PROP. D. B. XI.

It is very probable that Euclid gave this Propoſition a place in the Elements, ſince he gave the like Propoſition concerning equiangular parallelograms in the 23. B. 6.

PROP. XXXIV. B. XI.

In this the words, ὧν αἱ ἐφεϛῶσαι οὐκ εἰσὶν ἐπὶ τῶν αὐτῶν εὐθειῶν, "of which the inſiſting ſtraight lines are not in the ſame ſtraight "lines" are thrice repeated; but theſe words ought either to be left out, as they are by Clavius, or in place of them ought to be put "whether the inſiſting ſtraight lines be, or be not, in the ſame "ſtraight lines." for the other Caſe is without any reaſon excluded. alſo the words, ὧν τὰ ὕψη, "of which the altitudes" are twice put for ὧν αἱ ἐφεϛῶσαι, "of which the inſiſting ſtraight lines;" which is a plain miſtake. for the altitude is always at right angles to the baſe.

PROP. XXXV. B. XI.

The angles ABH, DEM are demonſtrated to be right angles in a ſhorter way than in the Greek; and in the ſame way ACH, DFM may be demonſtrated to be right angles. alſo the repetition of the ſame Demonſtration, which begins with "in the ſame manner," is left out, as it was probably added to the Text by ſome Editor; for the words, "in like manner we may demonſtrate" are not inſerted except when the Demonſtration is not given, or when it is ſomething different from the other, if it be given, as in Prop. 26. of this Book. Campanus has not this repetition.

We have given another Demonſtration of the Corollary, beſides the one in the Original, by help of which the following 36. Propoſition may be demonſtrated without the 35.

PROP. XXXVI. B. XI.

Tacquet in his Euclid demonſtrates this Propoſition without the help of the 35. but it is plain that the ſolids mentioned in the Greek Text in the Enuntiation of the Propoſition as equiangular, are ſuch that their ſolid angles are contained by three plane angles equal to one another, each to each; as is evident from the conſtruction. Now Tacquet does not demonſtrate, but aſſumes theſe ſolid angles to be equal to one another; for he ſuppoſes the ſolids to be already made, and does not give the conſtruction by which they are made. but, by the ſecond Demonſtration of the preceding Corollary, his Demonſtration is rendered legitimate likewiſe in the Caſe where the ſolids are conſtructed as in the Text.

PROP. XXXVII. B. XI.

In this it is aſſumed that the ratios which are triplicate of thoſe ratios which are the ſame with one another, are likewiſe the ſame with one another; and that thoſe ratios are the ſame with one another, of which the triplicate ratios are the ſame with one another; but this ought not to be granted without a Demonſtration, nor did Euclid aſſume the firſt and eaſieſt of theſe two Propoſitions, but demonſtrated it in the caſe of duplicate ratios, in the 22. Prop. B. 6. on this account another Demonſtration is given of this Propoſition like to that which Euclid gives in Prop. 22. B. 6. as Clavius has done.

PROP. XXXVIII. B. XI.

When it is required to draw a perpendicular from a point in one plane which is at right angles to another plane, unto this laſt plane, it is done by drawing a perpendicular from the point to the common ſection of the planes; for this perpendicular will be perpendicular to the plane, by Def. 4. of this Book. and it would be fooliſh in this caſe to do it by the 11. Propoſition of the ſame. but Euclid [a], Apollonius, and other Geometers, when they have occaſion for this Problem, direct a perpendicular to be drawn from the point to the

a. 17. 12. in other Editions.

Book XI.

plane, and conclude that it will fall upon the common ſection of the planes, becauſe this is the very ſame thing as if they had made uſe of the conſtruction above-mentioned, and then concluded that the ſtraight line muſt be perpendicular to the plane; but is expreſſed in fewer words. ſome Editor not perceiving this, thought it was neceſſary to add this Propoſition, which can never be of any uſe, to the 11. Book. and its being near to the end among Propoſitions with which it has no connexion, is a mark of its having been added to the Text.

PROP. XXXIX. B. XI.

In this it is ſuppoſed that the ſtraight lines which biſect the ſides of the oppoſite planes, are in one plane, which ought to have been demonſtrated; as is now done.

Book XII.

B. XII.

THE learned Mr. Moor, Profeſſor of Greek in the Univerſity of Glaſgow, obſerved to me that it plainly appears from Archimedes Epiſtle to Doſitheus prefixed to his Books of the Sphere and Cylinder, which Epiſtle he has reſtored from antient Manuſcripts, that Eudoxus was the Author of the chief Propoſitions in this twelfth Book.

PROP. II. B. XII.

At the beginning of this it is ſaid, "if it be not ſo, the ſquare of "BD ſhall be to the ſquare of FH, as the circle ABCD is to ſome "ſpace either leſs than the circle EFGH, or greater than it." and the like is to be found near to the end of this Propoſition, as alſo in Prop. 5. 11. 12. 18. of this Book. concerning which it is to be obſerved, that in the Demonſtration of Theorems, it is ſufficient, in this and the like caſes, that a thing made uſe of in the reaſoning can poſſibly exiſt, providing this be evident, tho' it cannot be exhibited or found by a Geometrical conſtruction. ſo in this place it is aſſumed that there may be a fourth proportional to theſe three magnitudes, viz. the ſquares of BD, FH, and the circle ABCD; becauſe it is evident that there is ſome ſquare equal to the circle ABCD, tho' it cannot be found geometrically; and to the three rectilineal figures, viz. the ſquares of BD, FH, and the ſquare which is equal to the circle ABCD, there is a fourth ſquare proportional; becauſe to the three ſtraight lines which are their ſides there is a fourth ſtraight line proportional[a], and this fourth ſquare, or a ſpace equal to it, is the ſpace which in this Propoſition is denoted by the letter S. and the like is to be underſtood in the other places above cited. and it is probable that this has been ſhewn by Euclid, but left out by ſome Editor; for the Lemma which ſome unſkilful hand has added to this Propoſition explains nothing of it.

a. 12. 6.

PROP. III. B. XII.

In the Greek Text and the Tranſlations, it is ſaid, "and be- "cauſe the two ſtraight lines BA, AC which meet one another" &c. here the angles BAC, KHL are demonſtrated to be equal to one another by 10. Prop. B. 11. which had been done before. becauſe the triangle EAG was proved to be ſimilar to the triangle KHL. this repetition is left out, and the triangles BAC, KHL are proved to be ſimilar in a ſhorter way by Prop. 21. B. 6.

PROP. IV. B. XII.

A few things in this are more fully explained than in the Greek Text.

PROP. V. B. XII.

In this, near to the end, are the words *ὡς ἔμπροσθεν ἐδείχθη*, "as "was before ſhewn," and the ſame are found again in the end of Prop. 18. of this Book; but the Demonſtration referred to, except it be the uſeleſs Lemma annexed to the 2. Prop. is no where in theſe Elements, and has been perhaps left out by ſome Editor who has forgot to cancel thoſe words alſo.

PROP. VI. B. XII.

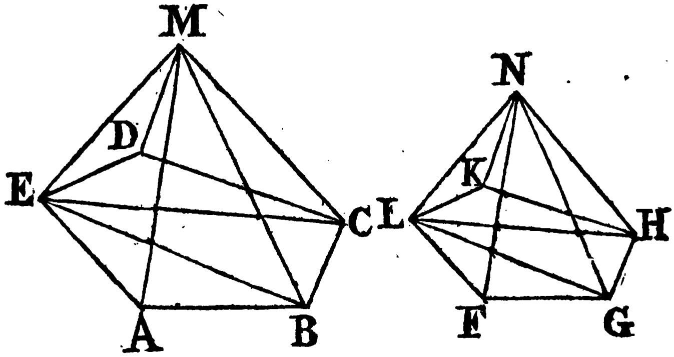
A ſhorter Demonſtration is given of this; and that which is in the Greek Text may be made ſhorter by a ſtep than it is. for the Author of it makes uſe of the 22. Prop. of B. 5. twice, whereas once would have ſerved his purpoſe; becauſe that Propoſition extends to any number of magnitudes which are proportionals taken two and two, as well as to three which are proportional to other three.

COR. PROP. VIII. B. XII.

The Demonſtration of this is imperfect, becauſe it is not ſhewn that the triangular pyramids into which thoſe upon multangular baſes are divided, are ſimilar to one another, as ought neceſſarily to have been done, and is done in the like caſe in Prop. 12. of this Book. the full Demonſtration of the Corollary is as follows.

Upon the polygonal baſes ABCDE, FGHKL, let there be ſimilar and ſimilarly ſituated pyramids which have the points M, N for their vertices. the pyramid ABCDEM has to the pyramid FGHKLN the triplicate ratio of that which the ſide AB has to the homologous ſide FG.

a. 20. 6. b. 11. Def. 11. c. 4. 6. Let the polygons be divided into the triangles ABE, EBC, ECD; FGL, LGH, LHK, which are ſimilar [a] each to each. and becauſe the pyramids are ſimilar, therefore [b] the triangle EAM is ſimilar to the triangle LFN, and the triangle ABM to FGN. wherefore [c] ME is to EA, as NL to LF; and as AE to EB, ſo is FL to LG, becauſe



the triangles EAB, LFG are ſimilar; therefore, ex aequali, as ME to EB, ſo is NL to LG. in like manner it may be ſhewn that EB is to BM, as LG to GN; therefore, again, ex aequali, as EM to MB, ſo is LN to NG. wherefore the triangles EMB, LNG having d. 5. 6. their ſides proportionals are [d] equiangular, and ſimilar to one another. therefore the pyramids which have the triangles EAB, LFG for their baſes, and the points M, N for their vertices are ſimilar [b] e. B. 11. to one another, for their ſolid angles are [e] equal, and the ſolids themſelves are contained by the ſame number of ſimilar planes. in the ſame manner the pyramid EBCM may be ſhewn to be ſimilar to the pyramid LGHN, and the pyramid ECDM to LHKN. and becauſe the pyramids EABM, LFGN are ſimilar, and have triangular baſes, the pyramid EABM has [f] to LFGN the triplicate ratio of that which EB has to the homologous ſide LG. and, in the ſame manner, the pyramid EBCM has to the pyramid LGHN the triplicate ratio of that which EB has to LG. therefore as the pyramid EABM is to the pyramid LFGN, ſo is the pyramid EBCM to the pyramid LGHN. in like manner, as the pyramid EBCM is to LGHN, ſo is the pyramid ECDM to the pyramid LHKN. and as one of the antecedents is to one of the conſequents, ſo are all the antecedents to all the conſequents. therefore as the pyramid EABM to the pyramid LFGN, ſo is the whole pyramid ABCDEM to the whole pyramid FGHKLN. and the pyramid EABM has to the pyramid LFGN the triplicate ratio of that which AB has to FG, therefore the whole pyramid has to the whole pyramid the triplicate ratio of that which AB has to the homologous ſide FG. Q. E. D.

[f] 8. 12.

PROP. XI. and XII. B. XII.

The order of the letters of the Alphabet is not obſerved in theſe two Propoſitions, according to Euclid's manner, and is now reſtored. by which means the firſt part of Prop. 12. may be demonſtrated in the ſame words with the firſt part of Prop. 11. on this account the Demonſtration of that firſt part is left out, and aſſumed from Prop. 11.

PROP. XIII. B. XII.

In this Propoſition the common ſection of a plane parallel to the baſes of a cylinder, with the cylinder itſelf is ſuppoſed to be a circle, and it was thought proper briefly to demonſtrate it; from whence it is ſufficiently manifeſt that this plane divides the cylinder into two others. and the ſame thing is underſtood to be ſupplied in Prop. 14.

PROP. XV. B. XII.

"And complete the cylinders AX, EO." both the Enuntiation and Expoſition of the Propoſition repreſent the cylinders as well as the cones as already deſcribed. wherefore the reading ought rather to be "and let the cones be ALC, ENG; and the cylinders AX, EO."

The firſt Caſe in the ſecond part of the Demonſtration is wanting; and ſomething alſo in the ſecond Caſe of that part, before the repetition of the conſtruction is mentioned; which are now added.

PROP. XVII. B. XII.

In the Enuntiation of this Propoſition the Greek words, *εἰς τὴν μείζονα σφαῖραν στερεὸν πολύεδρον ἐγγράψαι, μὴ ψαῦον τῆς ἐλάσσονος σφαίρας κατὰ τὴν ἐπιφάνειαν*, are thus tranſlated by Commandine and others, "in majori ſolidum polyhedrum deſcribere quod minoris "ſphaerae ſuperficiem non tangat;" that is, "to deſcribe in the "greater ſphere a ſolid polyhedron which ſhall not meet the ſuper-"ficies of the leſſer ſphere." whereby they refer the words *κατὰ τὴν ἐπιφάνειαν* to theſe next to them *τῆς ἐλάσσονος σφαίρας*. but they ought by no means to be thus tranſlated, for the ſolid polyhedron doth not only meet the ſuperficies of the leſſer ſphere, but pervades the whole of that ſphere. therefore the foreſaid words are to be referred to *τὸ στερεὸν πολύεδρον*, and ought thus to be tranſlated, viz. to deſcribe in the greater ſphere a ſolid polyhedron whoſe ſuperficies ſhall not meet the leſſer ſphere; as the meaning of the Propoſition neceſſarily requires.

The Demonſtration of the Propoſition is ſpoiled and mutilated. for ſome eaſy things are very explicitly demonſtrated, while others not ſo obvious are not ſufficiently explained; for example, when it is affirmed that the ſquare of KB is greater than the double of the ſquare of BZ, in the firſt Demonſtration; and that the angle BZK is obtuſe, in the ſecond. both which ought to have been demonſtrated. beſides, in the firſt Demonſtration it is ſaid "draw KΩ "from the point K perpendicular to BD;" whereas it ought to have been ſaid, "join KV," and it ſhould have been demonſtrated that KV is perpendicular to BD. for it is evident from the figure in Hervagius's and Gregory's Editions, and from the words of the Demonſtration, that the Greek Editor did not perceive that the perpendicular drawn from the point K to the ſtraight line BD muſt neceſſarily fall upon the point V, for in the figure it is made to fall upon the point Ω a different point from V, which is likewiſe ſuppoſed in the Demonſtration. Commandine ſeems to have been aware of this; for in his figure he marks one and the ſame point with the two letters V, Ω; and before Commandine, the learned John Dee in the Commentary he annexes to this Propoſition in Henry Billingſley's Tranſlation of the Elements printed at London Ann. 1570, expreſsly takes notice of this error, and gives a Demonſtration ſuited to the Conſtruction in the Greek Text, by which he ſhews that the perpendicular drawn from the point K to BD, muſt neceſſarily fall upon the point V.

Likewiſe it is not demonſtrated that the quadrilateral figures SOPT, TPRY, and the triangle YRX do not meet the leſſer ſphere, as was neceſſary to have been done. only Clavius, as far as I know, has obſerved this, and demonſtrated it by a Lemma, which is now premiſed to this Propoſition, ſomething altered and more briefly demonſtrated.

In the Corollary of this Propoſition it is ſuppoſed that a ſolid polyhedron is deſcribed in the other ſphere ſimilar to that which is deſcribed in the ſphere BCDE. but as the Conſtruction by which this may be done is not given, it was thought proper to give it, and to demonſtrate that the pyramids in it are ſimilar to thoſe of the ſame order in the ſolid polyhedron deſcribed in the ſphere BCDE.

From the preceding Notes it is ſufficiently evident how much the Elements of Euclid, who was a moſt accurate Geometer, have been vitiated and mutilated by ignorant Editors. The opinion which the greateſt part of learned men have entertained concerning the preſent Greek edition, viz. that it is very little or nothing different from the genuine work of Euclid, has, without doubt deceived them, and made them leſs attentive and accurate in examining that Edition; whereby ſeveral errors, ſome of them groſs enough, have eſcaped their notice from the age in which Theon lived to this time. Upon which account there is ſome ground to hope that the pains we have taken in correcting thoſe errors, and freeing the Elements as far as we could from blemiſhes, will not be unacceptable to good Judges who can diſcern when Demonſtrations are legitimate, and when they are not.

The objections which, ſince the firſt Edition, have been made againſt ſome things in the Notes, eſpecially againſt the doctrine of Proportionals, have either been fully anſwered in Dr. Barrow's Lect. Mathemat. and in theſe Notes; or are ſuch, except one which has been taken notice of in the Note on Prop. 1. Book 11. as ſhew that the perſon who made them has not ſufficiently conſidered the things againſt which they are brought; ſo that it is not neceſſary to make any further anſwer to theſe objections and others like them againſt Euclid's Definition of Proportionals, of which Definition Dr. Barrow juſtly ſays in page 297 of the above-named book, that "Niſi machinis impulſa validioribus aeternum perſiſtet "inconcuſſa."

FINIS.

EUCLID'S DATA.

IN THIS EDITION

SEVERAL ERRORS ARE CORRECTED,

AND

SOME PROPOSITIONS ADDED,

BY

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PREFACE.

EUCLID'S DATA is the firſt in order of the books written by the antient Geometers to facilitate and promote the method of Reſolution or Analyſis. In the general, a thing is ſaid to be given which is either actually exhibited, or can be found out, that is, which is either known by Hypotheſis, or that can be demonſtrated to be known; and the Propoſitions in the Book of Euclid's Data ſhew what things can be found out or known from thoſe that by Hypotheſis are already known; ſo that in the Analyſis or Inveſtigation of a Problem, from the things that are laid down to be known or given, by the help of theſe Propoſitions other things are demonſtrated to be given, and, from theſe, other things are again ſhewn to be given, and ſo on, until that which was propoſed to be found out in the Problem is demonſtrated to be given, and when this is done the Problem is ſolved, and its Compoſition is made and derived from the Compoſitions of the Data which were made uſe of in the Analyſis. And thus the Data of Euclid are of the moſt general and neceſſary uſe in the ſolution of Problems of every kind.

Euclid is reckoned to be the Author of the Book of the Data both by the antient and modern Geometers; and there ſeems to be no doubt of his having written a Book on this ſubject, but which in the courſe of ſo many ages has been much vitiated by unſkilful Editors in ſeveral places, both in the order of the Propoſitions, and in the Definitions and Demonſtrations themſelves. To correct the errors which are now found in it, and bring it nearer to the accuracy with which it was, no doubt, at firſt written by Euclid, is the deſign of this Edition, that ſo it may be rendered more uſeful to Geometers, at leaſt to beginners who deſire to learn the inveſtigatory method of the antients. And for their ſakes the Compoſition of moſt of the Data are ſubjoined to their Demonſtrations, that the Compoſitions of Problems ſolved by help of the Data may be the more eaſily made.

Marinus the Philoſopher's preface, which in the Greek Edition is prefixed to the Data, is here left out, as being of no uſe to underſtand them. at the end of it he ſays that Euclid has not uſed the ſynthetical, but the analytical method in delivering them; in which he is quite miſtaken; for in the Analyſis of a Theorem the thing to be demonſtrated is aſſumed in the Analyſis; but in the Demonſtrations of the Data, the thing to be demonſtrated, which is that ſomething or other is given, is never once aſſumed in the Demonſtration, from which it is manifeſt that every one of them is demonſtrated ſynthetically; tho' indeed if a Propoſition of the Data be turned into a Problem, for example the 84th or 85th in the former Editions, which here are the 85th and 86th, the Demonſtration of the Propoſition becomes the Analyſis of the Problem.

Wherein this Edition differs from the Greek, and the reaſons of the alterations from it will be ſhewn in the Notes at the end of the Data.

EUCLID'S DATA.

DEFINITIONS.

I.

SPACES, lines and angles are ſaid to be given in magnitude, when equals to them can be found.

II.

A ratio is ſaid to be given, when a ratio of a given magnitude to a given magnitude which is the ſame ratio with it can be found.

III.

Rectilineal figures are ſaid to be given in ſpecies, which have each of their angles given, and the ratios of their ſides given.

IV.

Points, lines and ſpaces are ſaid to be given in poſition, which have always the ſame ſituation, and which are either actually exhibited, or can be found.

A.

An angle is ſaid to be given in poſition, which is contained by ſtraight lines given in poſition.

V.

A circle is ſaid to be given in magnitude, when a ſtraight line from its center to the circumference is given in magnitude.

VI.

A circle is ſaid to be given in poſition and magnitude, the center of which is given in poſition, and a ſtraight line from it to the circumference is given in magnitude.

VII.

Segments of circles are ſaid to be given in magnitude, when the angles in them, and their baſes are given in magnitude.

VIII.

Segments of circles are ſaid to be given in poſition and magnitude, when the angles in them are given in magnitude, and their baſes are given both in poſition and magnitude.

IX.

A magnitude is ſaid to be greater than another by a given magnitude, when this given magnitude being taken from it, the remainder is equal to the other magnitude.

X.

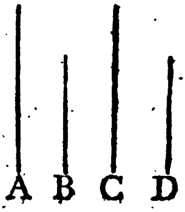
A magnitude is ſaid to be leſs than another by a given magnitude, when this given magnitude being added to it, the whole is equal to the other magnitude.

* 1. PROPOSITION I.

See N. THE ratio of given magnitudes to one another is given.

Let A, B be two given magnitudes, the ratio of A to B is given.

a. 1. Def. Dat. Becauſe A is a given magnitude, there may [a] be found one equal to it; let this be C. and becauſe B is given, one equal to it may be found; let it be D. and ſince b. 7. 5. A is equal to C, and B to D; therefore [b] A is to B, as C to D; and conſequently the ratio of A to B is given, becauſe the ratio of the given magnitudes C, D which is the ſame with it has been found.

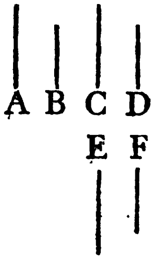


2. PROP. II.

See N. IF a given magnitude has a given ratio to another magnitude, "and if unto the two magnitudes by "which the given ratio is exhibited, and the given "magnitude, a fourth proportional can be found;" the other magnitude is given.

Let the given magnitude A have a given ratio to the magnitude B; if a fourth proportional can be found to the three magnitudes above-named, B is given in magnitude.

a. 1. Def. Becauſe A is given, a magnitude may be found equal to it [a]; let this be C. and becauſe the ratio of A to B is given, a ratio which is the ſame with it may be found; let this be the ratio of the given magnitude E to the given magnitude F. unto the magnitudes E, F, C find a fourth proportional D, which, by the Hypotheſis, can be done. wherefore becauſe A is to B, as E to F; and as E to F,



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* The figures in the margin ſhew the number of the Propoſitions in the other Editions.

ſo is C to D; A is [b] to B, as C to D. but A is equal to C, therefore [c] B is equal to D. the magnitude B is therefore given [a], becauſe a magnitude D equal to it has been found.

b. 11. 5.
c. 14. 5.

The limitation within the inverted commas is not in the Greek text, but is now neceſſarily added; and the ſame muſt be underſtood in all the Propoſitions of the Book which depend upon this ſecond Propoſition, where it is not expreſsly mentioned. See the Note upon it.

PROP. III.

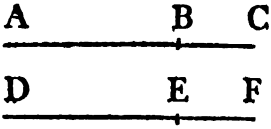
3.

IF any given magnitudes be added together, their ſum ſhall be given.

Let any given magnitudes AB, BC be added together, their ſum AC is given.

Becauſe AB is given, a magnitude equal to it may be found [a]; let this be DE. and becauſe BC is given, one equal to it may be found; let this be EF. wherefore becauſe AB is equal to DE, and BC equal to EF; the whole AC is equal to the whole DF. AC is therefore given, becauſe DF has been found, which is equal to it.

a. 1. Def.



PROP. IV.

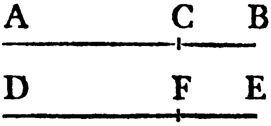
4.

IF a given magnitude be taken from a given magnitude; the remaining magnitude ſhall be given.

From the given magnitude AB let the given magnitude AC be taken; the remaining magnitude CB is given.

Becauſe AB is given, a magnitude equal to it may [a] be found; let this be DE. and becauſe AC is given, one equal to it may be found; let this be DF. wherefore becauſe AB is equal to DE, and AC to DF; the remainder CB is equal to the remainder FE. CB is therefore given [a], becauſe FE which is equal to it has been found.

a. 1. Def.

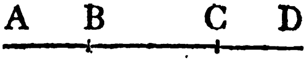


12. PROP. V.

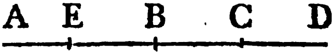
See N. IF of three magnitudes, the firſt together with the ſecond be given, and alſo the ſecond together with the third; either the firſt is equal to the third, or one of them is greater than the other by a given magnitude.

Let AB, BC, CD be three magnitudes, of which AB together with BC, that is AC, is given; and alſo BC together with CD, that is BD, is given. either AB is equal to CD, or one of them is greater than the other by a given magnitude.

Becauſe AC, BD are each of them given, they are either equal to one another, or not equal. firſt, let them be equal, and becauſe AC is equal to BD, take away the common part BC; therefore the remainder AB is equal to the remainder CD.



But if they be unequal, let AC be greater than BD, and make CE equal to BD. therefore CE is given, becauſe BD is given. and the whole AC is given, therefore
a. 4. Dat. [a] AE the remainder is given. and becauſe EC is equal to BD, by taking BC from both, the remainder EB is equal to the remainder CD. and AE is given, wherefore AB exceeds EB, that is CD by the given magnitude AE.

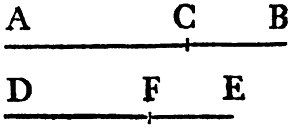


5. PROP. VI.

See N. IF a magnitude has a given ratio to a part of it; it ſhall alſo have a given ratio to the remaining part of it.

Let the magnitude AB have a given ratio to AC a part of it; it has alſo a given ratio to the remainder BC.

a. 2. Def. Becauſe the ratio of AB to AC is given, a ratio may be found [a] which is the ſame to it. let this be the ratio of DE a given magnitude to the given magnitude DF. and becauſe DE, DF are given, the remainder FE is [b] given. and becauſe AB is to
b. 4. Dat.
c. E. 5. AC, as DE to DF, by converſion [c] AB is to BC, as DE to EF. therefore the ratio of AB to BC is given, becauſe the ratio of the given magnitudes DE, EF which is the ſame with it has been found.



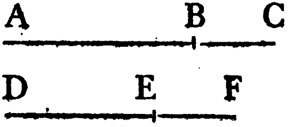
Cor. From this it follows, that the parts AC, CB have a given ratio to one another. becauſe as AB to BC, ſo is DE to EF; by diviſion [d], AC is to CB, as DF to FE; and DF, FE are given; d. 17. 5.
therefore [a] the ratio of AC to CB is given. a. 2. Def.

PROP. VII. 6.

IF two magnitudes which have a given ratio to one another, be added together; the whole magnitude ſhall have to each of them a given ratio. See N.

Let the magnitudes AB, BC which have a given ratio to one another, be added together; the whole AC has to each of the magnitudes AB, BC a given ratio.

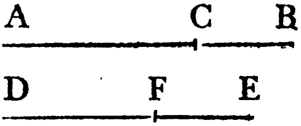
Becauſe the ratio of AB to BC is given, a ratio may be found [a] a. 2. Def.
which is the ſame with it; let this be the ratio of the given magnitudes DE, EF. and becauſe DE, EF are given, the whole DF is given [b]. b. 3. Dat.
and becauſe as AB to BC, ſo is DE to EF; by compoſition [c], AC is to CB, c. 18. 5.
as DF to FE; and by converſion [d], d. E. 5.
AC is to AB, as DF to DE. wherefore becauſe AC is to each of the magnitudes AB, BC, as DF to each of the others DE, EF; the ratio of AC to each of the magnitudes AB, BC is given [a].



PROP. VIII. 7.

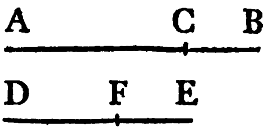
IF a given magnitude be divided into two parts which have a given ratio to one another, and if a fourth proportional can be found to the ſum of the two magnitudes by which the given ratio is exhibited, one of them, and the given magnitude; each of the parts is given. See N.

Let the given magnitude AB be divided into the parts AC, CB which have a given ratio to one another; if a fourth proportional can be found to the above-named magnitudes; AC and CB are each of them given.



Becauſe the ratio of AC to CB is given, the ratio of AB to BC is given [a]; therefore a ratio which is a. 7. Dat.

b. 2. Def. the ſame with it can be found[b], let this be the ratio of the given magnitudes DE, EF. and becauſe the given magnitude AB has to BC the given ratio of DE to EF, if unto DE, EF, AB a fourth proportional can be found, this c. 2. Dat. which is BC is given[c]; and becauſe AB d. 4. Dat. is given, the other part AC is given[d].



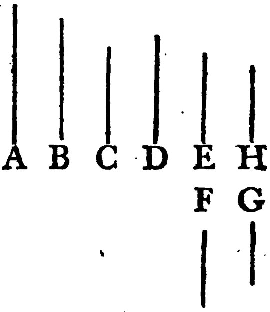
In the ſame manner, and with the like limitation, If the difference AC of two magnitudes AB, BC which have a given ratio be given; each of the magnitudes AB, BC is given.

8. PROP. IX.

MAGNITUDES which have given ratios to the ſame magnitude, have alſo a given ratio to one another.

Let A, C have each of them a given ratio to B; A has a given ratio to C.

Becauſe the ratio of A to B is given, a ratio which is the ſame a. 2. Def. to it may be found[a]; let this be the ratio of the given magnitudes D, E. and becauſe the ratio of B to C is given, a ratio which is the ſame with it may be found[a]; let this be the ratio of the given magnitudes F, G. to F, G, E find a fourth proportional H, if it can be done; and becauſe as A is to B, ſo is D to E; and as B to C, ſo is (F to G, and ſo is) E to H; ex aequali, as A to C, ſo is D to H. therefore the ratio of A to C is given[a], becauſe the ratio of the given magnitudes D and H, which is the ſame with it, has been found. but if a fourth proportional to F, G, E cannot be found, then it can only be ſaid that the ratio of A to C is compounded of the ratios of A to B, and B to C, that is of the given ratios of D to E, and F to G.



PROP. X. 9.

IF two or more magnitudes have given ratios to one another, and if they have given ratios, tho' they be not the ſame, to ſome other magnitudes: theſe other magnitudes ſhall alſo have given ratios to one another.

Let two or more magnitudes A, B, C have given ratios to one another; and let them have given ratios, tho' they be not the ſame, to ſome other magnitudes D, E, F. the magnitudes D, E, F have given ratios to one another.

Becauſe the ratio of A to B is given, and likewiſe the ratio of A to D; therefore the ratio of D to B is given [a]; but the ratio of B to E is given, therefore [a] the ratio of D to E is given. and becauſe the ratio of B to C is given, and alſo the ratio of B to E; the ratio of E to C is given [a]. and the ratio of C to F is given; wherefore the ratio of E to F is given. D, E, F have therefore given ratios to one another.

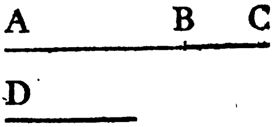
a. 5. Dat.

PROP. XI. 22.

IF two magnitudes have each of them a given ratio to another magnitude; both of them together ſhall have a given ratio to that other.

Let the magnitudes AB, BC have a given ratio to the magnitude D; AC has a given ratio to the ſame D.

Becauſe AB, BC have each of them a given ratio to D, the ratio of AB to BC is given [a]. and by compoſition, the ratio of AC to CB is given [b]. but the ratio of BC to D is given; therefore [a] the ratio of AC to D is given.



a. 9. Dat.
b. 7. Dat.

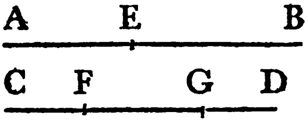
23.

PROP. XII.

See N. IF the whole have to the whole a given ratio, and the parts have to the parts given, but not the ſame, ratios, every one of them, whole or part, ſhall have to every one a given ratio.

Let the whole AB have a given ratio to the whole CD, and the parts AE, EB have given, but not the ſame, ratios to the parts CF, FD; every one ſhall have to every one, whole or part, a given ratio.

Becauſe the ratio of AE to CF is given, as AE to CF, ſo make AB to CG; the ratio therefore of AB to CG is given; wherefore the ratio of the remainder EB to the remainder FG is given, be-
a. 19. 5. cauſe it is the ſame[a] with the ratio of AB to CG. and the ratio of EB to FD is given, wherefore the ratio of
b. 9. Dat. FD to FG is given[b]; and by conver-
c. 6. Dat. ſion, the ratio of FD to DG is given[c]. and becauſe AB has to each of the magnitudes CD, CG a given ratio, the ratio of CD to CG is given[b]; and therefore[c] the ratio of CD to DG is given. but the ratio of GD to DF is given, wherefore[b] the ratio of CD to DF is given, and
d. Cor. 6. Dat. conſequently[d] the ratio of CF to FD is given; but the ratio of CF
e. 10. Dat. to AE is given, as alſo the ratio of FD to EB; wherefore[e] the ratio of AE to EB is given; as alſo the ratio of AB to each of
f. 7. Dat. them[f]. the ratio therefore of every one to every one is given.



24.

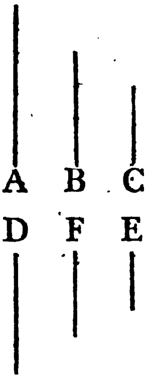
PROP. XIII.

See N. IF the firſt of three proportional ſtraight lines has a given ratio to the third, the firſt ſhall alſo have a given ratio to the ſecond.

Let A, B, C be three proportional ſtraight lines, that is as A to B, ſo is B to C; if A has to C a given ratio, A ſhall alſo have to B a given ratio.

Becauſe the ratio of A to C is given, a ratio which is the ſame
a. 2. Def. with it may be found[a]; let this be the ratio of the given ſtraight
b. 13. 6. lines D, E; and between D and E find a[b] mean proportional F;

therefore the rectangle contained by D and E is equal to the ſquare of F, and the rectangle D, E is given becauſe its ſides D, E are given; wherefore the ſquare of F, and the ſtraight line F is given. and becauſe as A is to C, ſo is D to E; but as A to C, ſo is [c] the ſquare of A to the ſquare of B; and as D to E, ſo is [c] the ſquare of D to the ſquare of F; therefore the ſquare [d] of A is to the ſquare of B, as the ſquare of D to the ſquare of F. as therefore [e] the ſtraight line A to the ſtraight line B, ſo is the ſtraight line D to the ſtraight line F. therefore the ratio of A to B is given [a], becauſe the ratio of the given ſtraight lines D, F which is the ſame with it has been found.



c. 2. Cor. 20. 6.
d. 11. 5.
e. 22. 6.
a. 2. Def.

PROP. XIV.

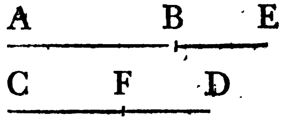
A.

IF a magnitude together with a given magnitude has a given ratio to another magnitude; the exceſs of this other magnitude above a given magnitude has a given ratio to the firſt magnitude. and if the exceſs of a magnitude above a given magnitude has a given ratio to another magnitude; this other magnitude together with a given magnitude has a given ratio to the firſt magnitude.

See N.

Let the magnitude AB together with the given magnitude BE, that is AE, have a given ratio to the magnitude CD; the exceſs of CD above a given magnitude has a given ratio to AB.

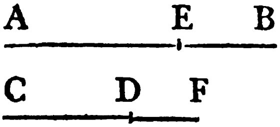
Becauſe the ratio of AE to CD is given, as AE to CD, ſo make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given [a]. and becauſe as AE to CD, ſo is BE to FD, the remainder AB is [b] to the remainder CF, as AE to CD. but the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is, CF the exceſs of CD above the given magnitude FD has a given ratio to AB.



a. 2. Dat.
b. 19. 5.

Next, Let the exceſs of the magnitude AB above the given magnitude BE, that is, let AE have a given ratio to the magnitude CD; CD together with a given magnitude has a given ratio to AB.

Becauſe the ratio of AE to CD is given, as AE to CD, ſo make
BE to FD; therefore the ratio of BE to
FD is given, and BE is given, wherefore
a. 2. Dat. FD is given [a]. and becauſe as AE to CD,
c. 12. 5. ſo is BE to FD, AB is to CF, as [c] AE to
CD. but the ratio of AE to CD is given,
therefore the ratio of AB to CF is given; that is CF which is equal
to CD together with the given magnitude DF has a given ratio to AB.



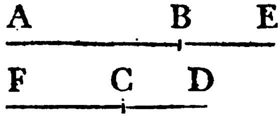
B.

PROP. XV.

See N. IF a magnitude together with that to which another magnitude has a given ratio, be given; this other is given together with that to which the firſt magnitude has a given ratio.

Let AB, CD be two magnitudes of which AB together with BE to which CD has a given ratio, is given; CD is given together with that magnitude to which AB has a given ratio.

Becauſe the ratio of CD to BE is given, as BE to CD, ſo make
AE to FD; therefore the ratio of AE to FD is given, and AE is
a. 2. Dat. given, wherefore [a] FD is given. and be-
cauſe as BE to CD, ſo is AE to FD;
b. Cor. 19. 5. AB is [b] to FC, as BE to CD. and the
ratio of BE to CD is given, wherefore
the ratio of AB to FC is given. and FD is given, that is CD together with FC to which AB has a given ratio is given.



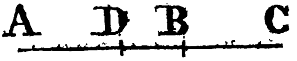
10.

PROP. XVI.

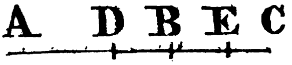
See N. IF the exceſs of a magnitude above a given magnitude, has a given ratio to another magnitude; the exceſs of both together above a given magnitude ſhall have to that other a given ratio. and if the exceſs of two magnitudes together above a given magnitude, has to one of them a given ratio; either the exceſs of the other above a given magnitude has to that one a given ratio; or the other is given together with the magnitude to which that one has a given ratio.

Let the exceſs of the magnitude AB above a given magnitude, have a given ratio to the magnitude BC; the exceſs of AC, both of them together, above a given magnitude, has a given ratio to BC.

Let AD be the given magnitude the exceſs of AB above which, viz. DB, has a given ratio to BC. and becauſe DB has a given ratio to BC, the ratio of DC to CB is given [a], and AD is given; therefore DC, the exceſs of AC above the given magnitude AD, has a given ratio to BC. a. 7. Dat.



Next, let the exceſs of two magnitudes AB, BC together above a given magnitude have to one of them BC a given ratio; either the exceſs of the other of them AB above a given magnitude ſhall have to BC a given ratio; or AB is given together with the magnitude to which BC has a given ratio.



Let AD be the given magnitude, and firſt let it be leſs than AB; and becauſe DC the exceſs of AC above AD has a given ratio to BC, DB has [b] a given ratio to BC; that is DB, the exceſs of AB above the given magnitude AD, has a given ratio to BC. b. Cor. 6. Dat.

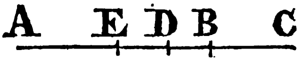
But let the given magnitude be greater than AB, and make AE equal to it; and becauſe EC, the exceſs of AC above AE, has to BC a given ratio, BC has [c] a given ratio to BE; and becauſe AE is given, AB together with BE to which BC has a given ratio, is given. c. 6. Dat.

PROP. XVII. 11.

IF the exceſs of a magnitude above a given magnitude has a given ratio to another magnitude; the exceſs of the ſame firſt magnitude above a given magnitude, ſhall have a given ratio to both the magnitudes together. and if the exceſs of either of two magnitudes above a given magnitude has a given ratio to both magnitudes together; the exceſs of the ſame above a given magnitude ſhall have a given ratio to the other. See N.

Let the exceſs of the magnitude AB above a given magnitude have a given ratio to the magnitude BC; the exceſs of AB above a given magnitude has a given ratio to AC.

Let AD be the given magnitude; and becauſe DB, the exceſs of
AB above AD, has a given ratio to BC; the ratio of DC to DB is
a. 7. Dat. given [a]. make the ratio of AD to DE the ſame with this ratio;
therefore the ratio of AD to DE is
given. and AD is given, wherefore
b. 2. Dat. [b] DE, and the remainder AE are
c. 12. 5. given. and becauſe as DC to DB, ſo is AD to DE, AC is [c] to EB,
as DC to DB; and the ratio of DC to DB is given, wherefore the
ratio of AC to EB is given. and becauſe the ratio of EB to AC
is given, and that AE is given, therefore EB the exceſs of AB
above the given magnitude AE, has a given ratio to AC.



Next, let the exceſs of AB above a given magnitude have a given ratio to AB and BC together, that is to AC; the exceſs of AB above a given magnitude has a given ratio to BC.

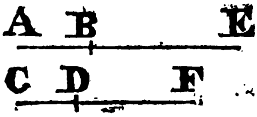
Let AE be the given magnitude; and becauſe EB the exceſs
of AB above AE has to AC a given ratio, as AC to EB, ſo make
d. 6. Dat. AD to DE; therefore the ratio of AD to DE is given, as alſo [d]
the ratio of AD to AE. and AE is given, wherefore [b] AD is given.
and becauſe as the whole, AC, to the whole, EB, ſo is AD
e. 19. 5. to DE; the remainder DC is [e] to the remainder DB, as AC to EB;
and the ratio of AC to EB is given, wherefore the ratio of DC
f. Cor. 6. Dat. to DB is given, as alſo [f] the ratio of DB to BC. and AD is given,
therefore DB, the exceſs of AB above the given magnitude
AD, has a given ratio to BC.

14. PROP. XVIII.

If to each of two magnitudes, which have a given ratio to one another, a given magnitude be added; the wholes ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

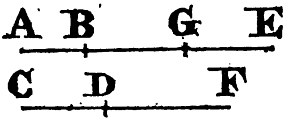
Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude BE be added, and the given magnitude DF to CD. the wholes AE, CF either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

a. 1. Dat. Becauſe BE, DF are each of them given, their ratio is given [a].



and if this ratio be the ſame with the ratio of AB to CD, the ratio of AE to CF, which is the ſame[b] with b. 12. 5. the given ratio of AB to CD, ſhall be given.

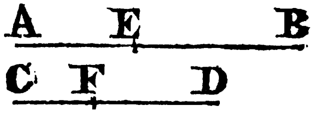
But if the ratio of BE to DF be not the ſame with the ratio of AB to CD; either it is greater than the ratio of AB to CD, or, by inverſion, the ratio of DF to BE is greater than the ratio of CD to AB. firſt, let the ratio of BE to DF be greater than the ratio of AB to CD; and as AB to CD, ſo make BG to DF; therefore the ratio of BG to DF is given; and DF is given, therefore[c] c. 2. Dat. BG is given. and becauſe BE has a greater ratio to DF than (AB to CD, that is than) BG to DF, BE is greater[d] than BG. and d. 10. 5. becauſe as AB to CD, ſo is BG to DF, therefore AG is[b] to CF, as AB to CD. but the ratio of AB to CD is given, wherefore the ratio of AG to CF is given; and becauſe BE, BG are each of them given, GE is given. therefore AG, the exceſs of AE above the given magnitude GE has a given ratio to CF. the other caſe is demonſtrated in the ſame manner.



PROP. XIX. 15.

IF from each of two magnitudes, which have a given ratio to one another, a given magnitude be taken; the remainders ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude, ſhall have a given ratio to the other.

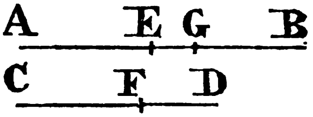
Let the magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and from CD, the given magnitude CF. the remainders EB, FD ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.



Becauſe AE, CF are each of them given, their ratio is given[a]; a. 1. Dat. and if this ratio be the ſame with the ratio of AB to CD, the ratio of the remainder EB to the re-

b. 19. 5. mainder FD, which is the ſame [b] with the given ratio of AB to CD, ſhall be given.

But if the ratio of AB to CD be not the ſame with the ratio of AE to CF, either it is greater than the ratio of AE to CF, or, by inverſion, the ratio of CD to AB is greater than the ratio of CF to AE. firſt, let the ratio of AB to CD be greater than the ratio of AE to CF, and as AB to CD, ſo make AG to CF; therefore the ratio of AG to CF is given, and CF
c. 2. Dat. is given, wherefore [c] AG is given. and becauſe the ratio of AB to CD, that is the ratio of AG to CF, is greater than the ratio of AE to CF;
d. 10. 5. AG is greater [d] than AE. and AG, AE are given, therefore the remainder EG is given. and as AB to CD, ſo is AG to CF, and ſo is [b] the remainder GB to the remainder FD; and the ratio of AB to CD is given, wherefore the ratio of GB to FD is given; therefore GB, the exceſs of EB above the given magnitude EG, has a given ratio to FD. in the ſame manner the other caſe is demonſtrated.



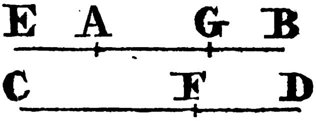
16.

PROP. XX.

IF to one of two magnitudes which have a given ratio to one another, a given magnitude be added, and from the other a given magnitude be taken; the exceſs of the ſum above a given magnitude ſhall have a given ratio to the remainder.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude EA be added, and from CD let the given magnitude CF be taken; the exceſs of the ſum EB above a given magnitude has a given ratio to the remainder FD.

Becauſe the ratio of AB to CD is given, make as AB to CD, ſo AG to CF. therefore the ratio of AG to CF is given, and CF is
a. 2. Dat. given, wherefore [a] AG is given; and EA is given, therefore the whole EG is given. and becauſe as AB to CD, ſo is AG to CF,
b. 19. 5. and ſo is [b] the remainder GB to the remainder FD; the ratio of GB to FD is given. and EG is given,



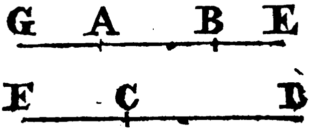
therefore GB, the exceſs of the ſum EB above the given magnitude EG, has a given ratio to the remainder FD.

PROP. XXI. C.

IF two magnitudes have a given ratio to one another, if See N. a given magnitude be added to one of them, and the other be taken from a given magnitude; the ſum together with the magnitude to which the remainder has a given ratio, is given. and the remainder is given together with the magnitude to which the ſum has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another; and to AB let the given magnitude BE be added, and let CD be taken from the given magnitude FD. the ſum AE is given together with the magnitude to which the remainder FC has a given ratio.

Becauſe the ratio of AB to CD is given, make as AB to CD, ſo GB to FD. therefore the ratio of GB to FD is given, and FD is given, wherefore GB is given [a]; a. 2. Dat. and BE is given, the whole GE is therefore given. and becauſe as AB to CD, ſo is GB to FD, and ſo is [b] GA to FC; b. 19. 5. the ratio of GA to FC is given. and AE together with GA is given, becauſe GE is given; therefore the ſum AE together with GA to which the remainder FC has a given ratio, is given. the ſecond part is manifeſt from Prop. 15.



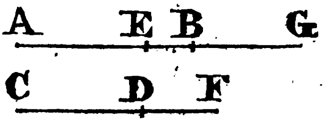
PROP. XXII. D.

IF two magnitudes have a given ratio to one another, See N. if from one of them a given magnitude be taken, and the other be taken from a given magnitude; each of the remainders is given together with the magnitude to which the other remainder has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and let

CD be taken from the given magnitude CF; the remainder EB is given together with the magnitude to which the other remainder DF has a given ratio.

Becauſe the ratio of AB to CD is given, make as AB to CD, ſo AG to CF. the ratio of AG to CF is therefore given, and CF is a. 2. Dat. given, wherefore [a] AG is given; and AE is given, and therefore the remainder EG is given. and becauſe as AB to CD, ſo is AG to b. 19. 1. CF, and ſo is [b] the remainder BG to the remainder DF; the ratio of BG to DF is given. and EB together with BG is given, becauſe EG is given. therefore the remainder EB together with BG to which DF the other remainder has a given ratio is given. the ſecond part is plain from Prop. 15.

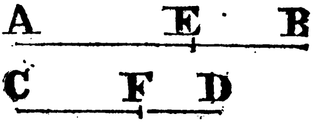


20. PROP. XXIII.

See N. IF from two given magnitudes there be taken magnitudes which have a given ratio to one another, the remainders ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

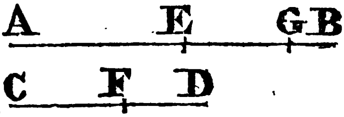
Let AB, CD be two given magnitudes, and from them let the magnitudes AE, CF which have a given ratio to one another be taken; the remainders EB, FD either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe AB, CD are each of them given, the ratio of AB to CD is given. and if this ratio be the ſame with the ratio of AE to a. 19. 5. CF, then the remainder EB has [a] the ſame given ratio to the remainder FD.



But if the ratio of AB to CD be not the ſame with the ratio of AE to CF, it is either greater than it, or, by inverſion, the ratio of CD to AB is greater than the ratio CF to AE. firſt, let the ratio of AB to CD be greater than the ratio of AE to CF; and as AE to CF, ſo make AG to CD, therefore the ratio of AG to CD is given, becauſe the ratio of AE to CF is given; and CD is given, where-

fore [b] AG is given; and becauſe the ratio of AB to CD is greater b. 2. Dat.
than the ratio of (AE to CF, that
is, than the ratio of) AG to CD;
AB is greater [c] than AG. and AB, c. 10. 5.
AG are given, therefore the remainder BG is given. and becauſe
as AE to CF, ſo is AG to CD, and ſo is [a] EG to FD; the ratio a. 19. 5.
of EG to FD is given. and GB is given, therefore EG the exceſs of EB above the given magnitude GB, has a given ratio to FD. the other caſe is ſhewn in the ſame way.

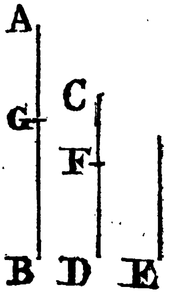


PROP. XXIV. 13.

IF there be three magnitudes, the firſt of which has a See N.
given ratio to the ſecond, and the exceſs of the ſecond above a given magnitude has a given ratio to the third; the exceſs of the firſt above a given magnitude ſhall alſo have a given ratio to the third.

Let AB, CD, E be three magnitudes, of which AB has a given ratio to CD; and the exceſs of CD above a given magnitude has a given ratio to E. the exceſs of AB above a given magnitude has a given ratio to E.

Let CF be the given magnitude the exceſs of CD above which, viz. FD has a given ratio to E. and becauſe the ratio of AB to CD is given, as AB to CD ſo make AG to CF; therefore the ratio of AG to CF is given; and CF is
given, wherefore [a] AG is given. and becauſe as a. 2. Dat.
AB to CD, ſo is AG to CF, and ſo is [b] GB to b. 19. 5.
FD; the ratio of GB to FD is given. and the
ratio of FD to E is given, wherefore [c] the ratio c. 9. Dat.
of GB to E is given. and AG is given, therefore GB the exceſs of AB above the given magnitude AG has a given ratio to E.



Cor. 1. And if the firſt has a given ratio to the ſecond, and the exceſs of the firſt above a given magnitude has a given ratio to the third; the exceſs of the ſecond above a given magnitude ſhall have a given ratio to the third: for if the ſecond be called the firſt, and the firſt the ſecond, this Corollary will be the ſame with the Propoſition.

Cor. 2. Alſo if the firſt has a given ratio to the ſecond, and the exceſs of the third above a given magnitude has alſo a given ratio to the ſecond, the ſame exceſs ſhall have a given ratio to the firſt; as is evident from the 9th Dat.

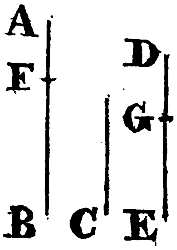
17. PROP. XXV.

If there be three magnitudes, the exceſs of the firſt whereof above a given magnitude has a given ratio to the ſecond; and the exceſs of the third above a given magnitude has a given ratio to the ſame ſecond. the firſt ſhall either have a given ratio to the third, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

Let AB, C, DE, be three magnitudes, and let the exceſſes of each of the two AB, DE above given magnitudes have given ratios to C; AB, DE either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Let FB the exceſs of AB above the given magnitude AF have a given ratio to C; and let GE the exceſs of DE above the given magnitude DG have a given ratio to C; and becauſe FB, GE have each of them a given ratio to C, they have a given ratio [a] to one another. but to FB, GE the given magnitudes AF, DG are added; therefore [b] the whole magnitudes AB, DE have either a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

a. 9. Dat.
b. 18. Dat.

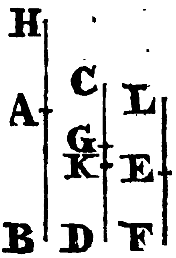


18. PROP. XXVI.

If there be three magnitudes the exceſſes of one of which above given magnitudes have given ratios to the other two magnitudes; theſe two ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

Let AB, CD, EF be three magnitudes, and let GD the exceſs of one of them CD above the given magnitude CG have a given ratio to AB; and alſo let KD the exceſs of the ſame CD above the given magnitude CK have a given ratio to EF. either AB has a given ratio to EF, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe GD has a given ratio to AB, as GD to AB, ſo make CG to HA; therefore the ratio of CG to HA is given; and CG is given, wherefore [a] HA is given. and becauſe as GD to AB, ſo is CG to HA, and ſo is [b] CD to HB; the ratio of CD to HB is given. alſo becauſe KD has a given ratio to EF, as KD to EF, ſo make CK to LE; therefore the ratio of CK to LE is given; and CK is given, wherefore [a] LE is given. and becauſe as KD to EF, ſo is CK to LE, and ſo [b] is CD to LF; the ratio of CD to LF is given. but the ratio of CD to HB is given, wherefore [c] the ratio of HB to LF is given. and from HB, LF the given magnitudes HA, LE being taken, the remainders AB, EF ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other [d].

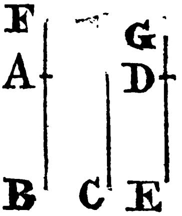


a. 2. Dat.
b. 12. 5.
c. 9. Dat.
d. 19. Dat.

" Another Demonſtration.

" Let AB, C, DE be three magnitudes, and let the exceſſes of one of them C above given magnitudes have given ratios to AB and DE. either AB, DE have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

" Becauſe the exceſs of C above a given magnitude has a given ratio to AB, therefore [a] AB together with a given magnitude has a given ratio to C. let this given magnitude be AF, wherefore FB has a given ratio to C. alſo, becauſe the exceſs of C above a given magnitude has a given ratio to DE, therefore [a] DE together with a given magnitude has a given ratio to C. let this given magnitude be DG, wherefore GE has a given ratio to C. and FB has a given ratio to C, therefore [b] the ratio of FB to GE is given. and from FB, GE the given magnitudes AF, DG being taken, the remainders AB, DE either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other [c]."



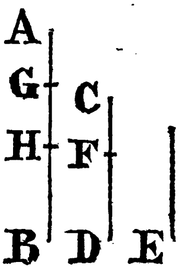
a. 14. Dat.
b. 9. Dat.
c. 19. Dat.

19. PROP. XXVII.

IF there be three magnitudes the exceſs of the firſt of which above a given magnitude has a given ratio to the ſecond: and the exceſs of the ſecond above a given magnitude has alſo a given ratio to the third. the exceſs of the firſt above a given magnitude ſhall have a given ratio to the third.

Let AB, CD, E be three magnitudes the exceſs of the firſt of which AB above the given magnitude AG, viz. GB has a given ratio to CD; and FD the exceſs of CD above the given magnitude CF, has a given ratio to E. the exceſs of AB above a given magnitude has a given ratio to E.

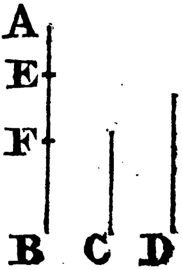
Becauſe the ratio of GB to CD is given, as GB to CD, ſo make
GH to CF; therefore the ratio of GH to CF is
a. 2. Dat. given; and CF is given, wherefore [a] GH is given;
and AG is given, wherefore the whole
AH is given. and becauſe as GB to CD, ſo is
b. 19. 5. GH to CF, and ſo is [b] the remainder HB to the
remainder FD; the ratio of HB to FD is given.
c. 9. Dat. and the ratio of FD to E is given, wherefore [c]
the ratio of HB to E is given. and AH is given;
therefore HB the exceſs of AB above the given
magnitude AH has a given ratio to E.



" Otherwiſe.

Let AB, C, D be three magnitudes, the exceſs EB of the firſt of which AB above the given magnitude AE has a given ratio to C, and the exceſs of C above a given magnitude has a given ratio to D. the exceſs of AB above a given magnitude has a given ratio to D.

Becauſe EB has a given ratio to C, and the
exceſs of C above a given magnitude has a given
d. 24. Dat. ratio to D; therefore [d] the exceſs of EB above a
given magnitude has a given ratio to D. let this
given magnitude be EF, therefore FB the exceſs of EB above EF has a given ratio to D.
and AF is given, becauſe AE, EF are given.



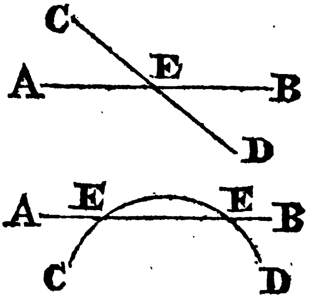
therefore FB the exceſs of AB above the given magnitude AF has a given ratio to D."

PROP. XXVIII.

25.

IF two lines given in poſition cut one another, the point or points in which they cut one another are given. See N.

Let two lines AB, CD given in poſition cut one another in the point E; the point E is given.



Becauſe the lines AB, CD are given in poſition, they have always the ſame ſituation [a], and therefore the point, or points, in which they cut one another have always the ſame ſituation, and becauſe the lines AB, CD can be found [a], the point, or points, in which they cut one another, are likewiſe found; and therefore are given in poſition [a].

a. 4. Def.

PROP. XXIX.

26.

IF the extremities of a ſtraight line be given in poſition; the ſtraight line is given in poſition and magnitude.

Becauſe the extremities of the ſtraight line are given, they can be found [a]; let theſe be the points A, B, between which a ſtraight line AB can be drawn [b]; this has an invariable poſition, becauſe between two given points there can be drawn but one ſtraight line. and when the ſtraight line AB is drawn, its magnitude is at the ſame time exhibited, or given. therefore the ſtraight line AB is given in poſition and magnitude.

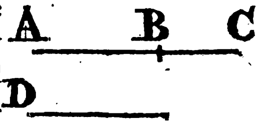
a. 4. Def.
b. 1. Poſtulate.

27. PROP. XXX.

IF one of the extremities of a ſtraight line given in poſition and magnitude be given; the other extremity ſhall alſo be given.

Let the point A be given, to wit, one of the extremities of a ſtraight line given in magnitude, and which lies in the ſtraight line AC given in poſition; the other extremity is alſo given.

Becauſe the ſtraight line is given in magnitude, one equal to it can be found [a]; let this be the ſtraight line D. from the greater ſtraight line AC cut off AB equal to the leſſer D. therefore the other extremity B of the ſtraight line AB is found. and the point B has always the ſame ſituation, becauſe any other point in AC, upon the ſame ſide of A, cuts off between it and the point A a greater or leſs ſtraight line than AB, that is than D. therefore the point B is given [b]. and it is plain another ſuch point can be found in AC produced upon the other ſide of the point A.



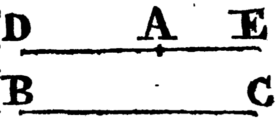
a. 1. Def.
b. 4. Def.

28. PROP. XXXI.

IF a ſtraight line be drawn through a given point parallel to a ſtraight line given in poſition; that ſtraight line is given in poſition.

Let A be a given point, and BC a ſtraight line given in poſition; the ſtraight line drawn thro' A parallel to BC is given in poſition.

Thro' A draw [a] the ſtraight line DAE parallel to BC; the ſtraight line DAE has always the ſame poſition, becauſe no other ſtraight line can be drawn through A parallel to BC. therefore the ſtraight line DAE which has been found is given [b] in poſition.

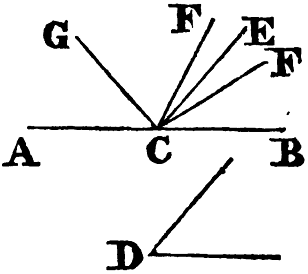


a. 31. 1.
b. 4. Def.

PROP. XXXII. 29.

IF a ſtraight line be drawn to a given point in a given ſtraight line, and makes a given angle with it. that ſtraight line is given in poſition.

Let AB be a ſtraight line given in poſition, and C a given point in it, the ſtraight line drawn to C which makes a given angle with CB, is given in poſition.



Becauſe the angle is given, one equal to it can be found [a]; let this be the angle at D. at the given point C in the given ſtraight line AB make [b] the angle ECB equal to the angle at D. therefore the ſtraight line EC has always the ſame ſituation, becauſe any other ſtraight line FC drawn to the point C makes with CB a greater or leſs angle than the angle ECB or the angle at D. therefore the ſtraight line EC which has been found is given in poſition.

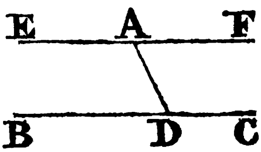
a. 1. Def.
b. 23. 1.

It is to be obſerved that there are two ſtraight lines EC, GC upon one ſide of AB that make equal angles with it, and which make equal angles with it when produced to the other ſide.

PROP. XXXIII. 30.

IF a ſtraight line be drawn from a given point, to a ſtraight line given in poſition, and makes a given angle with it; that ſtraight line is given in poſition.

From the given point A let the ſtraight line AD be drawn to the ſtraight line BC given in poſition, and make with it a given angle ADC; AD is given in poſition.



Thro' the point A draw [a] the ſtraight line EAF parallel to BC; and becauſe thro' the given point A the ſtraight line EAF is drawn parallel to BC which is given in poſition, EAF is therefore given in poſition [b]. and becauſe the ſtraight line AD meets the parallels BC, EF, the angle EAD is equal [c] to the angle ADC; and ADC is

a. 31. 1.
b. 31. Dat.
c. 29. 1.

given, wherefore alſo the angle EAD is given. therefore becauſe
the ſtraight line DA is drawn to the given point A in the ſtraight
line EF given in poſition, and makes with it a given angle EAD;
d. 32. Dat. AD is given[d] in poſition.

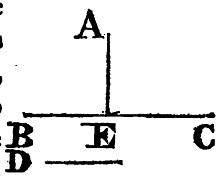
31.

PROP. XXXIV.

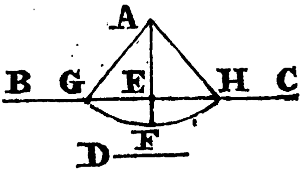
See N. IF from a given point to a ſtraight line given in poſition, a ſtraight line be drawn which is given in magnitude; the ſame is alſo given in poſition.

Let A be a given point, and BC a ſtraight line given in poſition; a ſtraight line given in magnitude drawn from the point A to BC is given in poſition.

Becauſe the ſtraight line is given in magnitude, one equal to it
a. 1. Def. can be found[a]; let this be the ſtraight line D. from the point A
draw AE perpendicular to BC; and becauſe AE is the ſhorteſt of all the ſtraight lines which can be drawn from the point A to BC, the ſtraight line D, to which one equal is to be drawn from the point A to BC, cannot be leſs than AE. If therefore D be equal to AE,
AE is the ſtraight line given in magnitude drawn from the given
b. 33. Dat. point A to BC. and it is evident that AE is given in poſition[b] becauſe it is drawn from the given point A to BC which is given in poſition, and makes with BC the given angle AEC.



But if the ſtraight line D be not equal to AE, it muſt be greater
than it. produce AE, and make AF equal to D; and from the center A, at the diſtance AF deſcribe the circle GFH, and join AG,
c. 6. Def. AH. becauſe the circle GFH is given in poſition[c], and the ſtraight
line BC is alſo given in poſition; therefore their interſection G is gi-
d. 28. Dat. ven[d]; and the point A is given;
e. 29. Dat. wherefore AG is given in poſition[e], that is, the ſtraight line AG given in magnitude (for it is equal to D) and drawn from the given point A to the ſtraight line BC given in poſition, is alſo given in poſition. and in like manner AH is given in poſition. therefore, in this caſe there are two ſtraight lines AG,



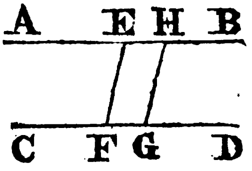
AH of the ſame given magnitude which can be drawn from a given point A to a ſtraight line BC given in poſition.

PROP. XXXV. 32.

IF a ſtraight line be drawn between two parallel ſtraight lines given in poſition, and makes given angles with them; the ſtraight line is given in magnitude.

Let the ſtraight line EF be drawn between the parallels AB, CD which are given in poſition, and make the given angles BEF, EFD; EF is given in magnitude.

In CD take the given point G, and thro' G draw [a] GH parallel to EF. and becauſe CD meets the parallels GH, EF, the angle EFD is equal [b] to the angle HGD. and EFD is a given angle, wherefore the angle HGD is given. and becauſe HG is drawn to the given point G in the ſtraight line CD given in poſition, and makes a given angle HGD; the ſtraight line HG is given in poſition [c]. and AB is given in poſition, therefore the point H is given [d]; and the point G is alſo given, wherefore GH is given in magnitude [e]. and EF is equal to it; therefore EF is given in magnitude.



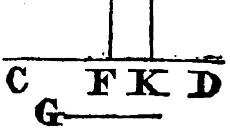
a. 31. 1.
b. 29. 1.
c. 32. Dat.
d. 28. Dat.
e. 29. Dat.

PROP. XXXVI. 33.

IF a ſtraight line given in magnitude be drawn between two parallel ſtraight lines given in poſition; it ſhall make given angles with the parallels. See N.

Let the ſtraight line EF given in magnitude be drawn between the parallel ſtraight lines AB, CD which are given in poſition; the angles AEF, EFC ſhall be given.

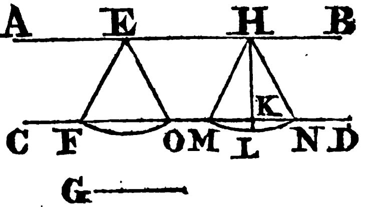
Becauſe EF is given in magnitude, a ſtraight line equal to it can be found [a]; let this be G. in AB take a given point H, and from it draw [b] HK perpendicular to CD. therefore the ſtraight line G, that is EF, cannot be leſs than HK.



a. 1. Def.
b. 12. 1.

and if G be equal to HK, EF alſo is equal to it; wherefore EF is at right angles to CD, for if it be not, EF would be greater than HK, which is abſurd. therefore the angle EFD is a right and conſequently a given angle.

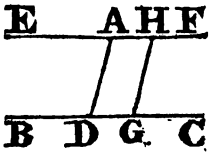
But if the ſtraight line G be not equal to HK, it muſt be greater than it. produce HK, and take HL equal to G; and from the center H, at the diſtance HL deſcribe the circle MLN, and join c. 6. Def. HM, HN. and becauſe the circle [c] MLN, and the ſtraight line CD d. 28. Dat. are given in poſition, the points M, N are [d] given; and the point H is given, wherefore the ſtraight lines HM, HN are given in poſi- e. 29. Dat. tion [e]. and CD is given in poſition, therefore the angles HMN, f. A. Def. HNM are given in poſition [f]. of the ſtraight lines HM, HN let HN be that which is not parallel to EF, for EF cannot be parallel to both of them; and draw EO parallel to HN. EO g. 34. 1. therefore is equal [g] to HN, that is to G; and EF is equal to G, wherefore EO is equal to EF, and the angle EFO to the angle h. 29. 1. EOF, that is [h] to the given angle HNM. and becauſe the angle HNM which is equal to the angle EFO or EFD has been found, therefore the angle EFD, that is the angle AEF, is given in mag- k. 1. Def. nitude [k], and conſequently the angle EFC.



E. PROP. XXXVII.

See N. IF a ſtraight line given in magnitude be drawn from a point to a ſtraight line given in poſition, in a given angle; the ſtraight line drawn thro' that point parallel to the ſtraight line given in poſition, is given in poſition.

Let the ſtraight line AD given in magnitude be drawn from the point A to the ſtraight line BC given in poſition, in the given angle ADC; the ſtraight line EAF drawn through A parallel to BC is given in poſition.



In BC take a given point G, and draw GH parallel to AD. and becauſe HG is drawn to a given point G in the ſtraight line BC given in poſition, in a given

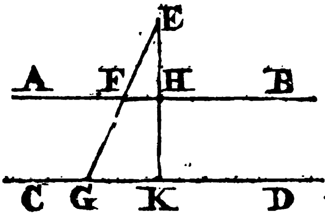
angle HGC, for it is equal [a] to the given angle ADC; HG is a. 29. 1.
given in poſition [b]; but it is given alſo in magnitude, becauſe it is b. 32. Dat.
equal to AD which is given in magnitude. therefore becauſe G one
of the extremities of the ſtraight line GH given in poſition and
magnitude is given, the other extremity H is given [c]. and the c. 30. Dat.
ſtraight line EAF which is drawn through the given point H
parallel to BC given in poſition, is therefore given [d] in poſition. d. 31. Dat.

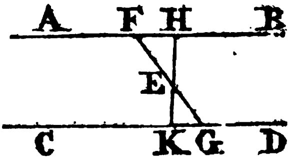
PROP. XXXVIII. 34.

IF a ſtraight line be drawn from a given point to two parallel ſtraight lines given in poſition; the ratio of the ſegments between the given point and the parallels ſhall be given.

Let the ſtraight line EFG be drawn from the given point E to the parallels AB, CD; the ratio of EF to EG is given.

From the point E draw EHK perpendicular to CD. and becauſe from a given point E the ſtraight line EK is drawn to CD which is given in poſition, in a given angle EKC; EK is given in poſi-





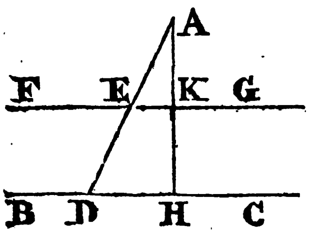
tion [a]. and AB, CD are given in poſition; therefore [b] the points a. 33. Dat.
H, K are given. and the point E is given, wherefore [c] EH, EK b. 28. Dat. c. 29. Dat.
are given in magnitude, and the ratio [d] of them is therefore given. d. 1. Dat.
but as EH to EK, ſo is EF to EG, becauſe AB, CD are parallels.
therefore the ratio of EF to EG is given.

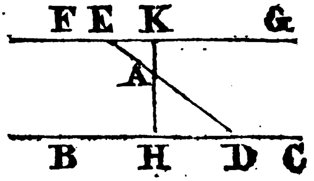
PROP. XXXIX. 35. 36.

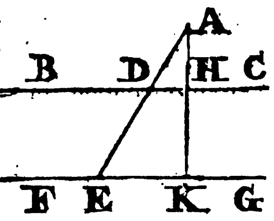
IF the ratio of the ſegments of a ſtraight line between a given point in it and two parallel ſtraight lines be given; if one of the parallels be given in poſition, the other is alſo given in poſition. See N.

From the given point A let the ſtraight line AED be drawn to the two parallel ſtraight lines FG, BC, and let the ratio of the ſegments AE, AD be given; if one of the parallels BC be given in poſition, the other FG is alſo given in poſition.

From the point A draw AH perpendicular to BC, and let it meet FG in K. and becauſe AH is drawn from the given point A to the ſtraight line BC given in poſition, and makes a given angle







a. 33. Dat. AHD; AH is given [a] in poſition. and
BC is likewiſe given in poſition, there-
b. 28. Dat. fore the point H is given [b]. the point A
is alſo given, wherefore AH is given in
c. 29. Dat. magnitude [c]. and, becauſe FG, BC are
parallels, as AE to AD, ſo is AK to
AH; and the ratio of AE to AD is gi-
ven, wherefore the ratio of AK to AH is given; but AH is given
d. 2. Dat. in magnitude, therefore [d] AK is given in magnitude; and it is alſo
e. 30. Dat. given in poſition, and the point A is given; wherefore [e] the point
K is given. and becauſe the ſtraight line FG is drawn thro' the
f. 31. Dat. given point K parallel to BC which is given in poſition, therefore [f]
FG is given in poſition.

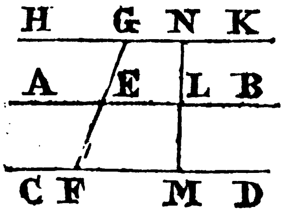
37. 38.

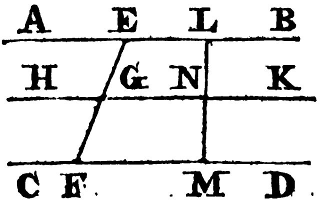
PROP. XL.

See N. IF the ratio of the ſegments of a ſtraight line into which it is cut by three parallel ſtraight lines, be given; If two of the parallels are given in poſition, the third alſo is given in poſition.

Let AB, CD, HK be three parallel ſtraight lines, of which AB, CD are given in poſition; and let the ratio of the ſegments GE, GF into which the ſtraight line GEF is cut by the three parallels, be given; the third parallel HK is given in poſition.

In AB take a given point L, and draw LM perpendicular to CD, meeting HK in N. becauſe LM is drawn from the given point L to CD which is given in poſition, and makes a given angle LMD; LM is given in poſition [a]. and CD is given in poſition, a. 33. Dat. wherefore the point M is given [b]; and the point L is given, LM b. 28. Dat. is therefore given in magnitude [c]. and becauſe the ratio of GE to c. 29. Dat.





GF is given, and as GE to GF, ſo is NL to NM; the ratio of NL to NM is given; and therefore [d] the ratio of ML to LN is d. Cor. 6. or 7. Dat. given. but LM is given in magnitude, wherefore [e] LN is given in e. 2. Dat. magnitude; and it is alſo given in poſition, and the point L is given; wherefore [f] the point N is given. and becauſe the ſtraight f. 30. Dat. line HK is drawn thro' the given point N parallel to CD which is given in poſition, therefore HK is given in poſition [g]. g. 31. Dat.

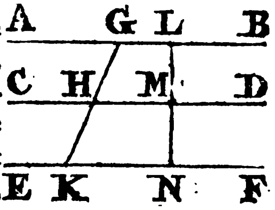
PROP. XLI.

F.

IF a ſtraight line meets three parallel ſtraight lines See N. which are given in poſition; the ſegments into which they cut it, have a given ratio.

Let the parallel ſtraight lines AB, CD, EF given in poſition be cut by the ſtraight line GHK; the ratio of GH to HK is given.

In AB take a given point L, and draw LM perpendicular to CD, meeting EF in N; therefore [a] LM is given in poſition; and CD, EF are given in poſition, wherefore the points M, N are given. and the point L is given, therefore [b] the ſtraight b. 29. Dat. lines LM, MN are given in magnitude; and the ratio of LM to MN is therefore



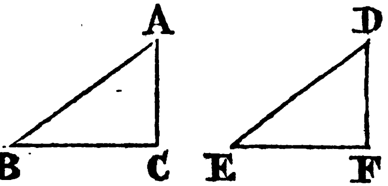
c. 1. Dat. given [c]. but as LM to MN, ſo is GH to HK; wherefore the
ratio of GH to HK is given.

39. PROP. XLII.

See N. IF each of the ſides of a triangle be given in magnitude; the triangle is given in ſpecies.

Let each of the ſides of the triangle ABC be given in magnitude; the triangle ABC is given in ſpecies.

a. 22. 1. Make a triangle [a] DEF the ſides of which are equal, each to
each, to the given ſtraight lines AB, BC, CA; which can be done,
becauſe any two of them muſt be greater than the third; and let
DE be equal to AB, EF to BC, and FD to CA. and becauſe the two ſides ED, DF are equal to the two BA, AC, each to each, and the baſe EF equal to the baſe BC; the angle EDF is



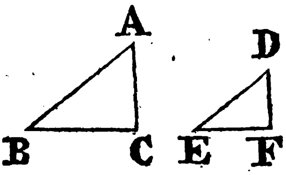
b. 8. 1. equal [b] to the angle BAC. therefore becauſe the angle EDF, which
is equal to the angle BAC, has been found, the angle BAC is gi-
c. 1. Def. ven [c], in like manner the angles at B, C are given. and becauſe the
d. 1. Dat. ſides AB, BC, CA are given, their ratios to one another are given[d].
e. 3. Def. therefore the triangle ABC is given [e] in ſpecies.

40. PROP. XLIII.

IF each of the angles of a triangle be given in magnitude; the triangle is given in ſpecies.

Let each of the angles of the triangle ABC be given in magnitude; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in
poſition and magnitude, and at the
a. 23. 1. points D, E make [a] the angle EDF e-
qual to the angle BAC, and the angle
DEF equal to ABC; therefore the
other angles EFD, BCA are equal.
and each of the angles at the points A, B, C is given, wherefore



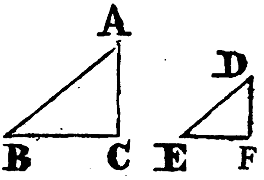
each of thoſe at the points D, E, F is given. and becauſe the ſtraight
line FD is drawn to the given point D in DE which is given in
poſition, making the given angle EDF; therefore DF is given in
poſition [b]. in like manner EF alſo is given in poſition; wherefore b. 32. Dat.
the point F is given. and the points D, E are given; therefore each
of the ſtraight lines DE, EF, FD is given [c] in magnitude. wherefore c. 29. Dat.
the triangle DEF is given in ſpecies [d]; and it is ſimilar [e] to the d. 42. Dat.
triangle ABC; which therefore is given in ſpecies. e. 4. 6. 1. Def. 6.

PROP. XLIV. 41.

IF one of the angles of a triangle be given, and if the ſides about it have a given ratio to one another; the triangle is given in ſpecies.

Let the triangle ABC have one of its angles BAC given, and let the ſides BA, AC about it have a given ratio to one another; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude, and
at the point D in the given ſtraight line DE made the angle EDF
equal to the given angle BAC; wherefore the angle EDF is given.
and becauſe the ſtraight line FD is drawn to the given point D in
ED which is given in poſition, making the given angle EDF; therefore
FD is given in poſition [a]. and be- a. 32. Dat.
cauſe the ratio of BA to AC is given,
make the ratio of ED to DF the ſame
with it, and join EF. and becauſe the
ratio of ED to DF is given, and ED is
given, therefore [b] DF is given in magnitude; b. 2. Dat.
and it is given alſo in poſition, and the point D is given,
wherefore the point F is given [c]. and the points D, E are given, c. 30. Dat.
wherefore DE, EF, FD are given [d] in magnitude; and the triangle d. 29. Dat.
DEF is therefore given [e] in ſpecies. and becauſe the triangles ABC, e. 42. Dat.
DEF have one angle BAC equal to one angle EDF, and the ſides
about theſe angles proportionals; the triangles are [f] ſimilar. but f. 6. 6.
the triangle DEF is given in ſpecies, and therefore alſo the triangle ABC.



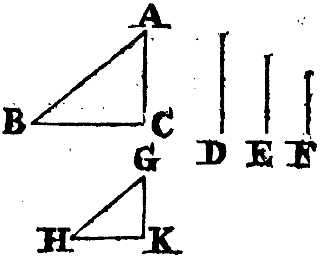
42.

PROP. XLV.

See N. **I**F the ſides of a triangle have to one another given ratios; the triangle is given in ſpecies.

Let the ſides of the triangle ABC have given ratios to one another. the triangle ABC is given in ſpecies.

Take a ſtraight line D given in magnitude; and becauſe the
ratio of AB to BC is given, make the ratio of D to E the ſame
a. 2. Dat. with it; and D is given, therefore [a] E is given. and becauſe the
ratio of BC to CA is given, to this make the ratio of E to F the
ſame; and E is given, and therefore [a] F. and becauſe as AB to
BC, ſo is D to E, by compoſition AB and BC together are to BC,
as D and E to E; but as BC to
CA, ſo is E to F; therefore, ex
b. 22. 5. aequali [b], as AB and BC are to
CA, ſo are D and E to F. and AB
c. 20. 1. and BC are greater [c] than CA,
d. A. 5. therefore D and E are greater [d]
than F. in the ſame manner any
two of the three D, E, F are
e. 22. 1. greater than the third. make [e] the
triangle GHK whoſe ſides are equal to D, E, F, ſo that GH be
equal to D, HK to E, and KG to F. and becauſe D, E, F are,
each of them, given, therefore GH, HK, KG are each of them
f. 42. Dat. given in magnitude; therefore the triangle GHK is given [f] in
ſpecies. but as AB to BC, ſo is (D to E, that is) GH to HK; and
as BC to CA, ſo is (E to F, that is) HK to KG; therefore, ex
g. 5. 6. aequali, as AB to AC, ſo is GH to GK. wherefore [g] the triangle
ABC is equiangular and ſimilar to the triangle GHK. and the
triangle GHK is given in ſpecies; therefore alſo the triangle
ABC is given in ſpecies.



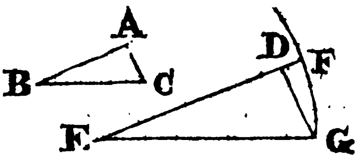
Cor. If a triangle is required to be made the ſides of which ſhall have the ſame ratios which three given ſtraight lines D, E, F have to one another; it is neceſſary that every two of them be greater than the third.

PROP. XLVI. 43.

IF the ſides of a right angled triangle about one of the acute angles have a given ratio to one another; the triangle is given in ſpecies.

Let the ſides AB, BC about the acute angle ABC of the triangle ABC which has a right angle at A, have a given ratio to one another; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude; and becauſe the ratio of AB to BC is given, make as AB to BC, ſo DE to EF; and becauſe DE has a given ratio to EF, and DE is given, therefore [a] EF is given. and becauſe as AB to BC, ſo is DE to EF, and AB is leſs [b] than BC, therefore DE is leſs [c] than EF. from the point D draw DG at right angles to DE, and from the center E at the diſtance EF deſcribe a circle which ſhall meet DG in two points, let G be either of them, and join EG; therefore the circumference of the circle is given [d] in poſition. and the ſtraight line DG is given [e] in poſition, becauſe it is drawn to the given point D in DE given in poſition, in a given angle. therefore [f] the point G is given. and the points D, E are given, wherefore DE, EG, GD are given [g] in magnitude, and the triangle DEG in ſpecies [h]. and becauſe the triangles ABC, DEG have the angle BAC equal to the angle EDG, and the ſides about the angles ABC, DEG proportionals, and each of the other angles BCA, EGD leſs than a right angle; the triangle ABC is equiangular [i] and ſimilar to the triangle DEG. but DEG is given in ſpecies, therefore the triangle ABC is given in ſpecies. and in the ſame manner, the triangle made by drawing a ſtraight line from E to the other point in which the circle meets DG is given in ſpecies.



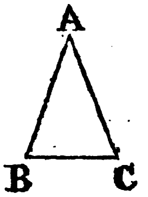
a. 2. Dat.
b. 19. 1.
c. A. 5.
d. 6. Def.
e. 32. Dat.
f. 28. Dat.
g. 29. Dat.
h. 42. Dat.
i. 7. 6.

44.

PROP. XLVII.

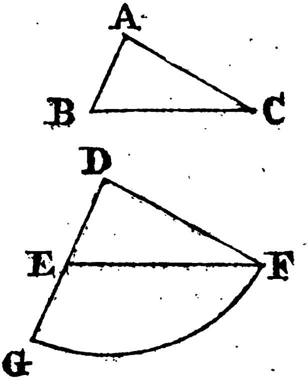
See N. IF a triangle has one of its angles which is not a right angle given, and if the ſides about another angle have a given ratio to one another; the triangle is given in ſpecies.

Let the triangle ABC have one of its angles ABC a given, but not a right angle, and let the ſides BA, AC about another angle BAC have a given ratio to one another; the triangle ABC is given in ſpecies.



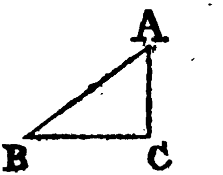
First, Let the given ratio be the ratio of equality, that is, let the ſides BA, AC and conſequently the angles ABC, ACB be equal. and becauſe the angle ABC is given, the angle ACB,
a. 32. 1. and alſo the remaining [a] angle BAC is given.
b. 43. Dat. therefore the triangle ABC is given [b] in ſpecies, and it is evident that in this caſe the given angle ABC muſt be acute.

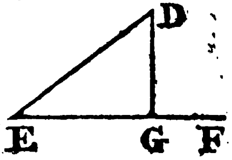
Next, Let the given ratio be the ratio of a leſs to a greater, that is, let the ſide AB adjacent to the given angle be leſs than the ſide AC. take a ſtraight line DE given in poſition and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF
c. 32. Dat. is given [c] in poſition. and becauſe the ratio of BA to AC is given, as BA to AC, ſo make ED to DG; and becauſe the ratio of ED to DG is given, and ED is given, the ſtraight
d. 2. Dat. line DG is given [d]. and BA is leſs
e. A. 5. than AC, therefore ED is leſs [e] than DG. from the center D, at the diſtance DG deſcribe the circle GF meeting EF in F, and join DF. and
f. 6. Def. becauſe the circle is given [f] in poſition, as alſo the ſtraight line EF, the
g. 28. Dat. point F is given [g]. and the points D, E are given, wherefore the
h. 29. Dat. ſtraight lines DE, EF, FD are given [h] in magnitude, and the tri-
i. 42. Dat. angle DEF in ſpecies [i]. and becauſe BA is leſs than AC, the angle
k. 18. 1. ACB is leſs [k] than the angle ABC, and therefore ACB is leſs



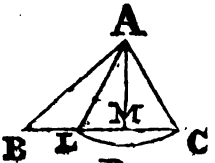
than a right angle. in the ſame manner, becauſe ED is leſs than DG or DF, the angle DFE is leſs than a right angle. l. 17. 1. and becauſe the triangles ABC, DEF have the angle ABC equal to the angle DEF, and the ſides about the angles BAC, EDF proportionals, and each of the other angles ACB, DFE leſs than a right angle; the triangles ABC, DEF are [m] ſimilar. m. 7. 6. and DEF is given in ſpecies, wherefore the triangle ABC is alſo given in ſpecies.

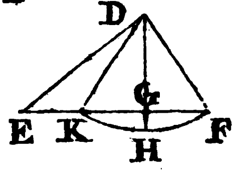
Thirdly, Let the given ratio be the ratio of a greater to a leſs, that is, let the ſide AB adjacent to the given angle be greater than AC. and, as in the laſt caſe, take a ſtraight line DE given in poſition and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given [c] in poſition. c. 32. Dat. alſo draw DG perpendicular to EF; therefore if the ratio of BA to AC be the ſame with the ratio of ED to the perpendicular DG, the triangles ABC, DEG are ſimilar [m], becauſe the angles ABC, DEG are equal, and DGE is a right angle. therefore the angle ACB is a right angle, and the triangle ABC is given [b] in ſpecies. b. 43. Dat.



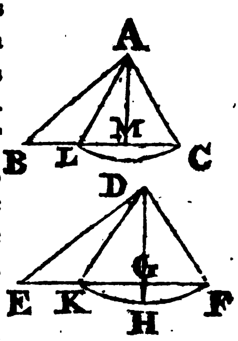


But if, in this laſt caſe, the given ratio of BA to AC be not the ſame with the ratio of ED to DG, that is, with the ratio of BA to the perpendicular AM drawn from A to BC; the ratio of BA to AC muſt be leſs [o] than the ratio of BA to AM, becauſe AC is o. 8. 5. greater than AM. make as BA to AC, ſo ED to DH; therefore the ratio of ED to DH is leſs than the ratio of (BA to AM, that is than the ratio of) ED to DG; and conſequently DH is greater [p] than DG; p. 10. 5. and becauſe BA is greater than AC, ED is greater [e] than DH. e. A. 5. from the center D, at the diſtance DH, deſcribe the circle KHF which neceſſarily meets the ſtraight line EF in two points, becauſe DH is greater than DG, and leſs than DE, let the circle meet EF in the points F, K which are given, as was ſhewn in the preceding caſe; and, DF, DK being joined, the triangles DEF, DEK are given in ſpecies, as was there ſhewn. from the center A, at the diſtance AC deſcribe a circle meeting BC again in L. and if the





angle ACB be leſs than a right angle, ALB muſt be greater than a right angle; and on the contrary. in the ſame manner, if the angle DFE be leſs than a right angle, DKE muſt be greater than one; and on the contrary. let each of the angles ACB, DFE be either leſs or greater than a right angle; and becauſe in the triangles ABC, DEF the angles ABC, DEF are equal, and the ſides BA, AC, and ED, DF about two of the other angles proportionals,
m. 7. 6. the triangle ABC is ſimilar [m] to the triangle
DEF. in the ſame manner, the triangle ABL is ſimilar to DEK. and the triangles DEF, DEK are given in ſpecies, therefore alſo the triangles ABC, ABL are given in ſpecies. and from this it is evident, that, in this third caſe, there are always two triangles of a different ſpecies to which the things mentioned as given in the Propoſition can agree.

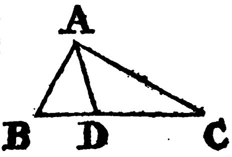


45. PROP. XLVIII.

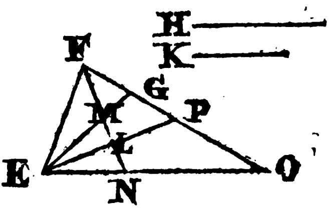
IF a triangle has one angle given, and if both the ſides together about that angle have a given ratio to the remaining ſide; the triangle is given in ſpecies.

Let the triangle ABC have the angle BAC given, and let the ſides BA, AC together about that angle have a given ratio to BC; the triangle ABC is given in ſpecies.

a. 9. 1. Biſect [a] the angle BAC by the ſtraight line AD; therefore the
b. 3. 6. angle BAD is given. and becauſe as BA to AC, ſo is [b] BD to DC,
by permutation, as AB to BD, ſo is AC to
CD; and as BA and AC together to BC, ſo
c. 12. 5. is [c] AB to BD. but the ratio of BA and AC
together to BC is given, wherefore the ratio
of AB to BD is given; and the angle BAD
d. 47. Dat. is given, therefore [d] the triangle ABD is gi-
ven in ſpecies. and the angle ABD is therefore given; the angle
e. 43. Dat. BAC is alſo given, wherefore the triangle ABC is given in ſpecies [e].



A triangle which ſhall have the things that are mentioned in the Propoſition to be given, can be found in the following manner. let EFG be the given angle, and let the ratio of H to K be the given ratio which the two ſides about the angle EFG muſt have to the third ſide of the triangle. therefore becauſe two ſides of a triangle are greater than the third ſide, the ratio of H to K muſt be the ratio of a greater to a leſs. biſect [a] the angle EFG by the a. 9. 1. ſtraight line FL, and by the 47th Propoſition find a triangle of which EFL is one of the angles, and in which the ratio of the ſides about the angle oppoſite to FL is the ſame with the ratio of H to K; to do which, take FE given in poſition and magnitude, and draw EL perpendicular to FL. then, if the ratio of H to K be the ſame with the ratio of FE to EL, produce EL and let it meet FG in P; the triangle FEP is that which was to be found. for it has the given angle EFG, and becauſe this angle is biſected by FL, the ſides EF, FP together are to EP, as [b] FE to EL, that is b. 3. 6. as H to K.



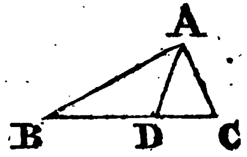
But if the ratio of H to K be not the ſame with the ratio of FE to EL, it muſt be leſs than it, as was ſhewn in Prop. 47. and in this caſe there are two triangles each of which has the given angle EFL, and the ratio of the ſides about the angle oppoſite to FL the ſame with the ratio of H to K. by Prop. 47. find theſe triangles EFM, EFN each of which has the angle EFL for one of its angles, and the ratio of the ſide FE to EM or EN the ſame with the ratio of H to K; and let the angle EMF be greater, and ENF leſs than a right angle. and becauſe H is greater than K, EF is greater than EN, and therefore the angle EFN, that is the angle NFG, is leſs [f] than the angle f. 18. 1 ENF. to each of theſe add the angles NEF, EFN; therefore the angles NEF, EFG are leſs than the angles NEF, EFN, FNE, that is than two right angles; therefore the ſtraight lines EN, FG muſt meet together when produced; let them meet in O, and produce EM to G. each of the triangles EFG, EFO has the things mentioned to be given in the Propoſition. for each of them has the given angle EFG, and becauſe this angle is biſected by the ſtraight line FMN, the ſides EF, FG together have to EG the third ſide the ratio of FE to EM, that is of H to K. in like manner, the ſides EF, FO together have to EO the ratio which H has to K.

46. PROP. XLIX.

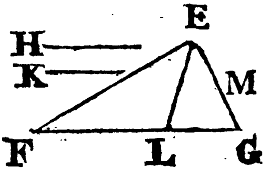
IF a triangle has one angle given, and if the ſides about another angle, both together, have a given ratio to the third ſide; the triangle is given in ſpecies.

Let the triangle ABC have one angle ABC given, and let the two ſides BA, AC about another angle BAC have a given ratio to BC; the triangle ABC is given in ſpecies.

Suppoſe the angle BAC to be biſected by the ſtraight line AD; BA and AC together are to BC, as AB to BD, as was ſhewn in the preceding Propoſition. but the ratio of BA and AC together to BC is given, therefore alſo the ratio of AB to BD is given. and
a. 44. Dat. the angle ABD is given, wherefore [a] the triangle ABD is given in ſpecies; and conſequently the angle BAD, and its double the angle BAC are given; and the angle ABC is given. therefore
b. 43. Dat. the triangle ABC is given in ſpecies [b].



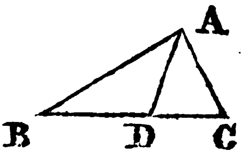
A triangle which ſhall have the things mentioned in the Propoſition to be given, may be thus found. Let EFG be the given angle, and the ratio of H to K the given ratio; and by Prop. 44. find the triangle EFL which has the angle EFG for one of its angles, and the ratio of the ſides EF, FL about this angle the ſame with the ratio of H to K; and make the angle LEM equal to the angle FEL. and becauſe the ratio of H to K is the ratio which two ſides of a triangle have to the third, H muſt be greater than K; and becauſe EF is to FL, as H to K, therefore EF is greater than FL, and the angle FEL, that is LEM is therefore leſs than the angle ELF. wherefore the angles LFE, FEM are leſs than two right angles, as was ſhewn in the foregoing Propoſition, and the ſtraight lines FL, EM muſt meet if produced; let them meet in G. EFG is the triangle which was to be found; for EFG is one of its angles, and becauſe the angle FEG is biſected by EL, the two ſides FE, EG together have to the third ſide FG the ratio of EF to FL, that is the given ratio of H to K.



PROP. L. 76.

IF from the vertex of a triangle given in ſpecies, a ſtraight line be drawn to the baſe in a given angle; it ſhall have a given ratio to the baſe.

From the vertex A of the triangle ABC which is given in ſpecies, let AD be drawn to the baſe BC in a given angle ADB; the ratio of AD to BC is given.



Becauſe the triangle ABC is given in ſpecies, the angle ABD is given, and the angle ADB is given; therefore the triangle ABD is given [a] in ſpecies; wherefore the ratio of AD to AB is given. and the ratio of AB to BC is given; and therefore [b] the ratio of AD to BC is given.

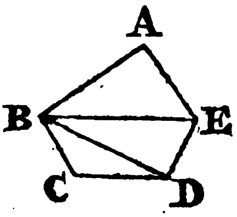
a. 43. Dat.

b. 9. Dat.

PROP. LI. 47.

RECTILINEAL figures given in ſpecies, are divided into triangles which are given in ſpecies.

Let the rectilineal figure ABCDE be given in ſpecies; ABCDE may be divided into triangles given in ſpecies.



Join BE, BD, and becauſe ABCDE is given in ſpecies, the angle BAE is given [a], and the ratio of BA to AE is given [a]; wherefore the triangle BAE is given in ſpecies [b], and the angle AEB is therefore given [a]. but the whole angle AED is given, and therefore the remaining angle BED is given. and the ratio of AE to EB is given, as alſo the ratio of AE to ED; therefore the ratio of BE to ED is given [c], and the angle BED is given, wherefore the triangle BED is given [b] in ſpecies. in the ſame manner the triangle BDC is given in ſpecies. therefore rectilineal figures which are given in ſpecies are divided into triangles given in ſpecies.

a. 3. Def.

b. 44. Dat.

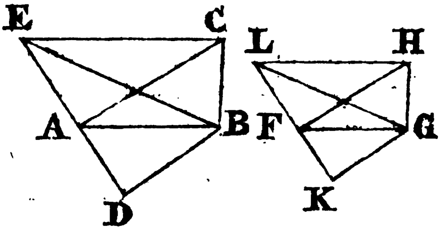
c. 9. Dat.

48. PROP. LII.

IF two triangles given in ſpecies be deſcribed upon the ſame ſtraight line; they ſhall have a given ratio to one another.

Let the triangles ABC, ABD given in ſpecies be deſcribed upon the ſame ſtraight line AB; the ratio of the triangle ABC to the triangle ABD is given.

Thro' the point C draw CE parallel to AB, and let it meet DA produced in E, and join BE. becauſe the triangle ABC is given in ſpecies, the angle BAC, that is the angle ACE, is given; and becauſe the triangle ABD is given in ſpecies, the angle DAB, that is
the angle AEC is given. therefore the triangle ACE is given in ſpecies; wherefore the ratio of EA to AC is
a. 3. Def. given [a], and the ratio of CA to AB is given, as alſo the ratio of BA
b. 9. Dat. to AD; therefore [b] the ratio of EA to AD is given. and the tri-
c. 37. 1. angle ACB is equal [c] to the triangle AEB, and as the triangle
d. 1. 6. AEB, or ACB, is to the triangle ADB, ſo is [d] the ſtraight line
EA to AD. but the ratio of EA to AD is given, therefore the ratio of the triangle ACB to the triangle ADB is given.



PROBLEM.

To find the ratio of two triangles ABC, ABD given in ſpecies, and which are deſcribed upon the ſame ſtraight line AB.

Take a ſtraight line FG given in poſition and magnitude, and becauſe the angles of the triangles ABC, ABD are given, at the
e. 23. 1. points F, G of the ſtraight line FG make [e] the angles GFH, GFK
equal to the angles BAC, BAD; and the angles FGH, FGK equal to the angles ABC, ABD, each to each. therefore the triangles ABC, ABD are equiangular to the triangles FGH, FGK, each to each. through the point H draw HL parallel to FG meeting KF produced in L. and becauſe the angles BAC, BAD are equal to the angles GFH, GFK, each to each; therefore the angles ACE, AEC are equal to FHL, FLH, each to each, and the triangle AEC equiangular to the triangle FLH. therefore as EA to AC,

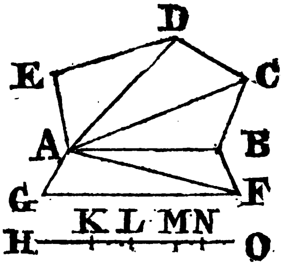
ſo is LF to FH; and as CA to AB, ſo HF to FG; and as BA to AD, ſo GF to FK; wherefore, ex aequali, as EA to AD, ſo is LF to FK. but, as was ſhewn, the triangle ABC is to the triangle ABD, as the ſtraight line EA to AD, that is as LF to FK. the ratio therefore of LF to FK has been found which is the ſame with the ratio of the triangle ABC to the triangle ABD.

PROP. LIII. 49.

IF two rectilineal figures given in ſpecies be deſcribed upon the ſame ſtraight line; they ſhall have a given ratio to one another. See N.

Let any two rectilineal figures ABCDE, ABFG which are given in ſpecies, be deſcribed upon the ſame ſtraight line AB; the ratio of them to one another is given.

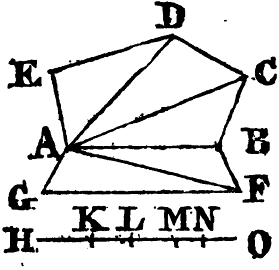
Join AC, AD, AF; each of the triangles AED, ADC, ACB, AGF, ABF is given [a] in ſpecies. and becauſe the triangles ADE, a. 51. Dat.
ADC given in ſpecies are deſcribed upon the ſame ſtraight line AD, the ratio of EAD to DAC is given [b]; and, b. 52. Dat.
by compoſition, the ratio of EACD to DAC is given [c]. and the ratio of DAC c. 7. Dat.
to CAB is given [b], becauſe they are deſcribed upon the ſame ſtraight line AC; therefore the ratio of EACD to ACB is given [d]; and, by compoſition, d. 9. Dat.
the ratio of ABCDE to ABC is given. in the ſame manner, the ratio of ABFG to ABF is given. but the ratio of the triangle ABC to the triangle ABF is given [b]; wherefore becauſe the ratio of ABCDE to ABC is given, as alſo the ratio of ABC to ABF, and the ratio of ABF to ABFG; the ratio of the rectilineal ABCDE to the rectilineal ABFG is given [d].



PROBLEM.

To find the ratio of two rectilineal figures given in ſpecies, and deſcribed upon the ſame ſtraight line.

Let ABCDE, ABFG be two rectilineal figures given in ſpecies, and deſcribed upon the ſame ſtraight line AB, and join AC, AD, AF. take a ſtraight line HK given in poſition and magnitude, and by the 52. Dat. find the ratio of the triangle ADE to the triangle ADC, and make the ratio of HK to KL the ſame with it. find alſo the ratio of the triangle ACD to the triangle ACB, and make the ratio of KL to LM the ſame. alſo, find the ratio of the triangle ABC to the triangle ABF, and make the ratio of LM to MN the ſame. and laſtly, find the ratio of the triangle AFB to the triangle AFG, and make the ratio of MN to NO the ſame. then the ratio of ABCDE to ABFG is the ſame with the ratio of HM to MO.

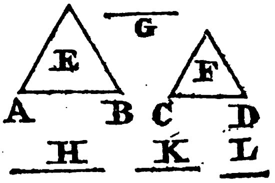


Becauſe the triangle EAD is to the triangle DAC, as the ſtraight line HK to KL; and as triangle DAC to CAB, ſo is the ſtraight line KL to LM; therefore by uſing compoſition as often as the number of triangles requires, the rectilineal ABCDE is to the triangle ABC, as the ſtraight line HM to ML. in like manner, becauſe triangle GAF is to FAB, as ON to NM, by compoſition, the rectilineal ABFG is to the triangle ABF, as MO to MN; and, by inverſion, as ABF to ABFG, ſo is NM to MO. and the triangle ABC is to ABF, as LM to MN. wherefore becauſe as ABCDE to ABC, ſo is HM to ML; and as ABC to ABF, ſo is LM to MN; and as ABF to ABFG, ſo is MN to MO; ex aequali, as the rectilineal ABCDE to ABFG, ſo is the ſtraight line HM to MO.

50. PROP. LIV.

IF two ſtraight lines have a given ratio to one another; the ſimilar rectilineal figures deſcribed upon them ſimilarly, ſhall have a given ratio to one another.

Let the ſtraight lines AB, CD have a given ratio to one another, and let the ſimilar and ſimilarly placed rectilineal figures E, F be deſcribed upon them; the ratio of E to F is given.



To AB, CD let G be a third proportional; therefore as AB to CD, ſo is CD to G. and the ratio of AB to CD is given, wherefore the ratio of CD to G is given; and conſequently the ratio of AB to G is alſo given [a]. but as AB to G, ſo is the figure E to the figure [b] F. therefore the ratio of E to F is given.

a. 9. Dat.
b. 2. Cor. 20. 6.

PROBLEM.

To find the ratio of two ſimilar rectilineal figures E, F ſimilarly deſcribed upon ſtraight lines AB, CD which have a given ratio to one another. let G be a third proportional to AB, CD.

Take a ſtraight line H given in magnitude; and becauſe the ratio of AB to CD is given, make the ratio of H to K the ſame with it; and becauſe H is given, K is given. as H is to K, ſo make K to L; then the ratio of E to F is the ſame with the ratio of H to L. for AB is to CD, as H to K, wherefore CD is to G, as K to L; and, ex aequali, as AB to G, ſo is H to L. but the figure E is to [b] the figure F, as AB to G, that is as H to L. b. 2. Cor. 20. 6.

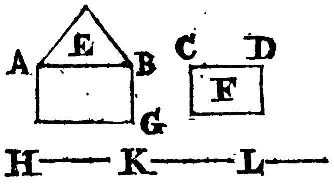
PROP. LV.

51.

IF two ſtraight lines have a given ratio to one another; the rectilineal figures given in ſpecies deſcribed upon them, ſhall have to one another a given ratio.

Let AB, CD be two ſtraight lines which have a given ratio to one another; the rectilineal figures E, F given in ſpecies and deſcribed upon them, have a given ratio to one another.

Upon the ſtraight line AB deſcribe the figure AG ſimilar and ſimilarly placed to the figure F; and becauſe F is given in ſpecies, AG is alſo given in ſpecies. therefore ſince the figures E, AG which are given in ſpecies, are deſcribed upon the ſame ſtraight line AB, the ratio of E to AG is given [a]. a. Dat. 53. and becauſe the ratio of AB to CD is given, and upon them are deſcribed the ſimilar and ſimilarly placed rectilineal figures AG, F, the ratio of AG to F is given [b]. and the ratio of AG to E is given; b. 54. Dat. therefore the ratio of E to F is given [c]. c. 9. Dat.



PROBLEM.

To find the ratio of two rectilineal figures E, F given in ſpecies, and deſcribed upon the ſtraight lines AB, CD which have a given ratio to one another.

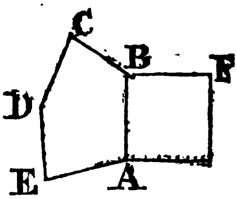
Take a ſtraight line H given in magnitude; and becauſe the rectilineal figures E, AG given in ſpecies are deſcribed upon the ſame ſtraight line AB, find their ratio by the 53. Dat. and make the ratio of H to K the ſame; K is therefore given. and becauſe the ſimilar rectilineal figures AG, F are deſcribed upon the ſtraight lines AB, CD which have a given ratio, find their ratio by the 54. Dat. and make the ratio of K to L the ſame. the figure E has to F the ſame ratio which H has to L. for, by the conſtruction, as E is to AG, ſo is H to K; and as AG to F, ſo is K to L; therefore, ex aequali, as E to F, ſo is H to L.

52.
PROP. LVI.

IF a rectilineal figure given in ſpecies be deſcribed upon a ſtraight line given in magnitude; the figure is given in magnitude.

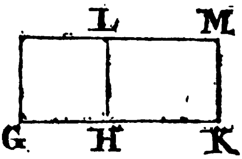
Let the rectilineal figure ABCDE given in ſpecies be deſcribed upon the ſtraight line AB given in magnitude; the figure ABCDE is given in magnitude.

Upon AB let the ſquare AF be deſcribed; therefore AF is given in ſpecies and magnitude. and becauſe the rectilineal figures ABCDE, AF given in ſpecies are deſcribed upon the ſame ſtraight line AB, the ratio of
a. 53. Dat. ABCDE to AF is given[a]. but the ſquare AF
b. 2. Dat. is given in magnitude, therefore [b] alſo the figure ABCDE is given in magnitude.



PROB.

To find the magnitude of a rectilineal figure given in ſpecies deſcribed upon a ſtraight line given in magnitude.



Take the ſtraight line GH equal to the given ſtraight line AB, and by the 53. Dat. find the ratio which the ſquare AF upon AB has to the figure ABCDE; and make the ratio of GH to HK the ſame; and upon GH deſcribe the ſquare GL, and complete the parallelogram LHKM; the figure ABCDE is equal to LHKM. becauſe AF is to ABCDE, as the ſtraight line GH to HK, that is, as the figure GL to HM; and AF is equal to GL, therefore
c. 14. 5. ABCDE is equal to HM [c].

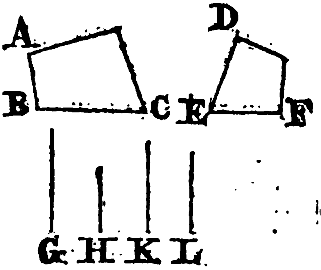
53.
PROP. LVII.

IF two rectilineal figures are given in ſpecies, and if a ſide of one of them has a given ratio to a ſide of the other; the ratios of the remaining ſides to the remaining ſides ſhall be given.

Let AC, DF be two rectilineal figures given in ſpecies, and let the ratio of the ſide AB to the ſide DE be given; the ratios of the remaining ſides to the remaining ſides are alſo given.

Becauſe the ratio of AB to DE is given, as alſo [a] the ratios of AB to BC, and of DE to EF; the ratio of BC to EF is given [b]. in the ſame manner, the ratios of the other ſides to the other ſides are given.

a. 3. Def.
b. 10. Dat.



The ratio which BC has to EF may be found thus; take a ſtraight line G given in magnitude; and becauſe the ratio of BC to BA is given, make the ratio of G to H the ſame; and becauſe the ratio of AB to DE is given, make the ratio of H to K the ſame; and make the ratio of K to L the ſame with the given ratio of DE to EF. ſince therefore as BC to BA, ſo is G to H; and as BA to DE, ſo is H to K; and as DE to EF, ſo is K to L; ex aequali, BC is to EF, as G to L. therefore the ratio of G to L has been found which is the ſame with the ratio of BC to EF.

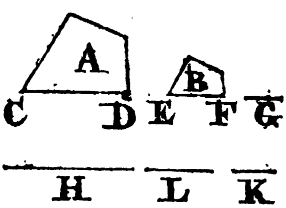
PROP. LVIII.

IF two ſimilar rectilineal figures have a given ratio to one another; their homologous ſides have alſo a given ratio to one another. See N.

Let the two ſimilar rectilineal figures A, B have a given ratio to one another; their homologous ſides have alſo a given ratio.

Let the ſide CD be homologous to EF, and to CD, EF let the ſtraight line G be a third proportional. as therefore [a] CD to G, ſo is the figure A to B; and the ratio of A to B is given, therefore the ratio of CD to G is given; and CD, EF, G are proportionals, wherefore [b] the ratio of CD to EF is given.

a. 2. Cor. 20. 6.
b. 13. Dat.



The ratio of CD to EF may be found thus; take a ſtraight line H given in magnitude; and becauſe the ratio of the figure A to B is given, make the ratio of H to K the ſame with it: and, as the 13. Dat. directs to be done, find a mean proportional L between H and K; the ratio of CD to EF is the ſame with that of H to L. let G be a third proportional to CD, EF; therefore as CD to G, ſo is (A to B, and ſo is) H to K. and as CD to EF, ſo is H to L, as is ſhewn in the 13. Dat.

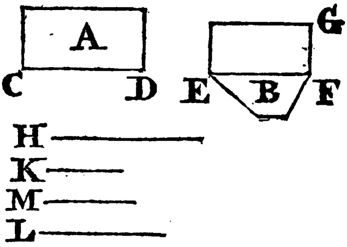
54. PROP. LIX.

See N. IF two rectilineal figures given in ſpecies have a given ratio to one another; their ſides ſhall likewiſe have given ratios to one another.

Let the two rectilineal figures A, B given in ſpecies have a given ratio to one another; their ſides ſhall alſo have given ratios to one another.

If the figure A be ſimilar to B, their homologous ſides ſhall have a given ratio to one another, by the preceding Propoſition; and becauſe the figures are given in ſpecies, the ſides of each of
a. 5. Def. them have given ratios [a] to one another. therefore each ſide of
b. 9. Dat. one of them has [b] to each ſide of the other a given ratio.

But if the figure A be not ſimilar to B, let CD, EF be any two of their ſides; and upon EF conceive the figure EG to be deſcribed ſimilar and ſimilarly placed to the figure A, ſo that CD, EF be homologous ſides; therefore EG is given in ſpecies. and the figure B is given in ſpecies,
c. 53. Dat. wherefore [c] the ratio of B to EG is given; and the ratio of A to B is given, therefore [b] the ratio of the figure A to EG is
d. 58. Dat. given. and A is ſimilar to EG, therefore [d] the ratio of the ſide CD to EF is given; and conſequently [b] the ratios of the remaining ſides to the remaining ſides are given.



The ratio of CD to EF may be found thus; take a ſtraight line H given in magnitude, and becauſe the ratio of the figure A to B is given, make the ratio of H to K the ſame with it. and by the 53. Dat. find the ratio of the figure B to EG, and make the ratio of K to L the ſame; between H and L find a mean proportional M; the ratio of CD to EF is the ſame with the ratio of H to M. becauſe the figure A is to B, as H to K; and as B to EG, ſo is K to L; ex aequali, as A to EG, ſo is H to L. and the figures A, EG are ſimilar, and M is a mean proportional between H and L; therefore, as was ſhewn in the preceding Propoſition, CD is to EF, as H to M.

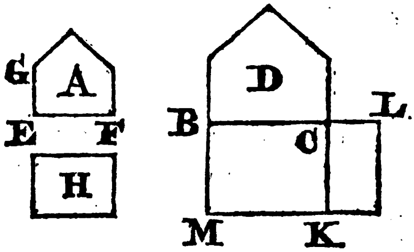
PROP. LX.

55.

IF a rectilineal figure be given in ſpecies and magnitude, the ſides of it ſhall be given in magnitude.

Let the rectilineal figure A be given in ſpecies and magnitude; its ſides are given in magnitude.

Take a ſtraight line BC given in poſition and magnitude; and upon BC deſcribe[a] the figure D ſimilar, and ſimilarly placed, to the figure A, and let EF be the ſide of the figure A homologous to BC the ſide of D; therefore the figure D is given in ſpecies. and becauſe upon the given ſtraight line BC the figure D given in ſpecies is deſcribed, D is given[b] in magnitude. and the figure A is given in magnitude, therefore the ratio of A to D is given. and the figure A is ſimilar to D; therefore the ratio of the ſide EF to the homologous ſide BC is given[c]. and BC is given, wherefore[d] EF is given. and the ratio of EF to EG is given[e], therefore EG is given. and, in the ſame manner, each of the other ſides of the figure A can be ſhewn to be given.



a. 18. 6.
b. 56. Dat.
c. 58. Dat.
d. 2. Dat.
e. 3. Def.

PROBLEM.

To deſcribe a rectilineal figure A ſimilar to a given figure D, and equal to another given figure H. It is Prop. 25. B. 6. Elem.

Becauſe each of the figures D, H is given, their ratio is given, which may be found by making[f] upon the given ſtraight line BC the parallelogram BK equal to D, and upon its ſide CK making[f] the parallelogram KL equal to H in the angle KCL equal to the angle MBC. therefore the ratio of D to H, that is of BK to KL is the ſame with the ratio of BC to CL. and becauſe the figures D, A are ſimilar, and that the ratio of D to A, or H, is the ſame with the ratio of BC to CL; by the 58. Dat. the ratio of the homologous ſides BC, EF is the ſame with the ratio of BC to the mean proportional between BC and CL. find EF the mean proportional; then EF is the ſide of the figure to be deſcribed, homologous to BC the

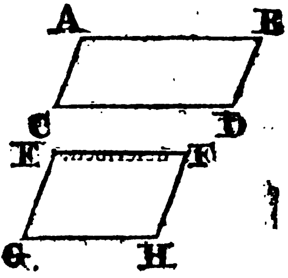
f. Cor. 45. 1.

ſide of D, and the figure itſelf can be deſcribed by the 18th Prop.
B. 6. which, by the conſtruction, is ſimilar to D. and becauſe D is
g. 2. Cor. 20. 6. to A, as [g] BC to CL, that is as the figure BK to KL; and that D
h. 14. 5. is equal to BK, therefore [h] A is equal to KL, that is to H.

57. PROP. LXI.

See N. IF a parallelogram given in magnitude has one of its ſides and one of its angles given in magnitude; the other ſide alſo is given.

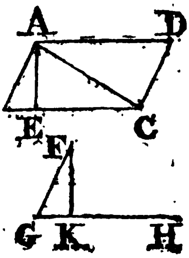
Let the parallelogram ABCD given in magnitude, have the ſide AB and the angle BAC given in magnitude; the other ſide AC is given.



Take a ſtraight line EF given in poſition and magnitude; and
becauſe the parallelogram AD is given
in magnitude, a rectilineal figure equal
a. 1. Def. to it can be found [a]. and a parallelo-
gram equal to this figure can be ap-
b. Cor. 45. 1. plied [b] to the given ſtraight line EF in
an angle equal to the given angle BAC.
let this be the parallelogram EFGH
having the angle FEG equal to the
angle BAC. and becauſe the parallelo-
grams AD, EH are equal, and have the angles at A and E equal;
c. 14. 6. the ſides about them are reciprocally proportional [c]. therefore as
AB to EF, ſo is EG to AC; and AB, EF, EG are given, therefore
d. 12. 6. alſo AC is given [d]. Whence the way of finding AC is manifeſt.

H. PROP. LXII.

See N. IF a parallelogram has a given angle, the rectangle contained by the ſides about that angle has a given ratio to the parallelogram.



Let the parallelogram ABCD have the given angle ABC; the rectangle AB, BC has a given ratio to the parallelogram AC.

From the point A draw AE perpendicular
to BC. becauſe the angle ABC is given, as
alſo the angle AEB; the triangle ABE is gi-
a. 43. Dat. ven [a] in ſpecies; therefore the ratio of BA to
b. 1. 6. AE is given. but as BA to AE, ſo is [b] the

rectangle AB, BC to the rectangle AE, BC; therefore the ratio of the rectangle AB, BC to AE, BC, that is [c] to the parallelogram AC is given. c. 35. 1.

And it is evident how the ratio of the rectangle to the parallelogram may be found, by making the angle FGH equal to the given angle ABC, and drawing, from any point F in one of its ſides, FK perpendicular to the other GH; for GF is to FK, as BA to AE, that is, as the rectangle AB, BC to the parallelogram AC.

Cor. And if a triangle ABC has a given angle ABC, the rectangle AB, BC contained by the ſides about that angle, ſhall have a given ratio to the triangle ABC. 66.

Complete the parallelogram ABCD; therefore, by this Propoſition, the rectangle AB, BC has a given ratio to the parallelogram AC; and AC has a given ratio to its half the triangle [d] ABC. therefore the rectangle AB, BC has a given [e] ratio to the triangle ABC. d. 41. 1. e. 9. Dat.

And the ratio of the rectangle to the triangle is found thus; make the triangle FGK as was ſhewn in the Propoſition; the ratio of GF to the half of the perpendicular FK is the ſame with the ratio of the rectangle AB, BC to the triangle ABC. becauſe, as was ſhewn, GF is to FK, as AB, BC to the parallelogram AC; and FK is to its half, as AC is to its half which is the triangle ABC; therefore, ex aequali, GF is to the half of FK, as AB, BC rectangle is to the triangle ABC.

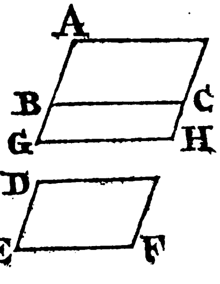
PROP. LXIII. 56.

If two parallelograms be equiangular, as a ſide of the firſt to a ſide of the ſecond, ſo is the other ſide of the ſecond to the ſtraight line to which the other ſide of the firſt has the ſame ratio which the firſt parallelogram has to the ſecond. and conſequently if the ratio of the firſt parallelogram to the ſecond be given, the ratio of the other ſide of the firſt to that ſtraight line is given; and if the ratio of the other ſide of the firſt to that ſtraight line be given, the ratio of the firſt parallelogram to the ſecond is given.

Let AC, DF be two equiangular parallelograms, as BC a ſide of the firſt is to EF a ſide of the ſecond, ſo is DE the other ſide of the ſecond to the ſtraight line to which AB the other ſide of the firſt has the ſame ratio which AC has to DF.

Produce the ſtraight line AB, and make as BC to EF, ſo DE to BG, and complete the parallelogram BGHC. therefore becauſe BC, or GH is to EF, as DE to BG, the ſides about the equal angles BGH, DEF are reciprocally proportional; wherefore [a] the parallelogram BH is equal to DF. and AB is to BG, as the parallelogram AC is to BH, that is to DF. as therefore BC is to EF, ſo is DE to BG, which is the ſtraight line to which AB has the ſame ratio that AC has to DF.

a. 14. 6.



And if the ratio of the parallelogram AC to DF be given, then the ratio of the ſtraight line AB to BG is given; and if the ratio of AB to the ſtraight line BG be given, the ratio of the parallelogram AC to DF is given.

74. 73.

PROP. LXIV.

See N.

IF two parallelograms have unequal, but given, angles, and if as a ſide of the firſt to a ſide of the ſecond, ſo the other ſide of the ſecond be made to a certain ſtraight line; if the ratio of the firſt parallelogram to the ſecond be given, the ratio of the other ſide of the firſt to that ſtraight line ſhall be given. and if the ratio of the other ſide of the firſt to that ſtraight line be given, the ratio of the firſt parallelogram to the ſecond ſhall be given.

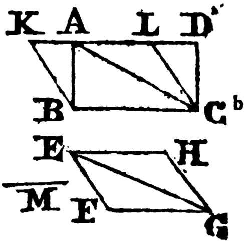
Let ABCD, EFGH be two parallelograms which have the unequal, but given, angles ABC, EFG; and as BC to FG, ſo make EF to the ſtraight line M. if the ratio of the parallelogram AC to EG be given, the ratio of AB to M is given.

At the point B of the ſtraight line BC make the angle CBK equal to the angle EFG, and complete the parallelogram KBCL. and becauſe the ratio of AC to EG is given, and that AC is equal [a] to the parallelogram KC, therefore the ratio of KC to EG is given, and KC, EG are equiangular; therefore as BC to FG, ſo is [b] EF to the ſtraight line to which KB has a given ratio, viz. the ſame which the parallelogram KC has to EG. but as BC to FG, ſo is EF to the ſtraight line M; therefore KB has a given ratio to M,

a. 35. 1.
b. 63. Dat.

and the ratio of AB to BK is given, becauſe the triangle ABK is given in ſpecies [c]. therefore the ratio of AB to M is given [d]. c. 43. Dat.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given. d. 9. Dat. for ſince the ratio of KB to BA is given, as alſo the ratio of AB to M, the ratio of KB to M is given [d]. and becauſe the parallelograms KC, EG are equiangular, as BC to FG, ſo is [b] EF to the ſtraight line b. 63. Dat. to which KB has the ſame ratio which the parallelogram KC has to EG. but as BC to FG, ſo is EF to M. therefore KB is to M, as the parallelogram KC is to EG. and the ratio of KB to M is given, therefore the ratio of the parallelogram KC, that is of AC to EG is given.



75. Cor. And if two triangles ABC, EFG have two equal angles, or two unequal, but given, angles ABC, EFG, and if as BC a ſide of the firſt to FG a ſide of the ſecond, ſo the other ſide of the ſecond EF be made to a ſtraight line M; if the ratio of the triangles be given, the ratio of the other ſide of the firſt to the ſtraight line M is given.

Complete the parallelograms ABCD, EFGH; and becauſe the ratio of the triangle ABC to the triangle EFG is given, the ratio of the parallelogram AC to EG is given [e], becauſe the parallelograms are double [f] of the triangles. e. 15. 5. f. 41. 1. and becauſe BC is to FG, as EF to M, the ratio of AB to M is given by the 63. Dat. if the angles ABC, EFG are equal; but if they be unequal, but given angles, the ratio of AB to M is given by this Propoſition.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given by the ſame Propoſitions; and therefore the ratio of the triangle ABC to EFG is given.

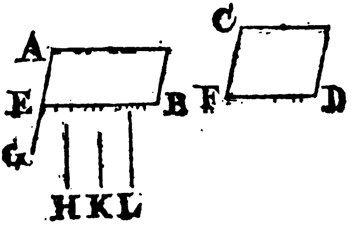
PROP. LXV.

68.

If two equiangular parallelograms have a given ratio to one another, and if one ſide has to one ſide a given ratio; the other ſide ſhall alſo have to the other ſide a given ratio.

Let the two equiangular parallelograms AB, CD have a given ratio to one another, and let the ſide EB have a given ratio to the ſide FD; the other ſide AE has alſo a given ratio to the other ſide CF.

Becauſe the two equiangular parallelograms AB, CD have a given ratio to one another; as EB a ſide of the firſt is to FD a ſide
a. 65. Dat. of the ſecond, ſo is [a] FC the other ſide of the ſecond to the ſtraight line to which AE the other ſide of the firſt has the ſame given ratio which the firſt parallelogram AB has to the other CD. let this ſtraight line be EG; therefore the ratio of AE to EG is given. and EB is to FD, as FC to EG, therefore the ratio of FC to EG is given, becauſe the ratio of EB to FD is given. and becauſe the ratio of AE to EG, as alſo the ratio of FC to EG is given; the
b. 9. Dat. ratio of AE to CF is given [b].



The ratio of AE to CF may be found thus; take a ſtraight line H given in magnitude, and becauſe the ratio of the parallelogram AB to CD is given, make the ratio of H to K the ſame with it. and becauſe the ratio of FD to EB is given, make the ratio of K to L the ſame. the ratio of AE to CF is the ſame with the ratio of H to L. make as EB to FD, ſo FC to EG, therefore, by inverſion, as FD to EB, ſo is EG to FC. and as AE to EG, ſo is [a] (the parallelogram AB to CD, and ſo is) H to K; but as EG to FC, ſo is (FD to EB, and ſo is) K to L; therefore, ex æquali, as AE to FC, ſo is H to L.

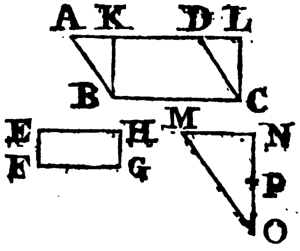
69. PROP. LXVI.

IF two parallelograms have unequal, but given, angles, and a given ratio to one another; if one ſide has to one ſide a given ratio, the other ſide has alſo a given ratio to the other ſide.

Let the two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG, have a given ratio to one another, and let the ratio of BC to FG be given; the ratio alſo of AB to EF is given.

At the point B of the ſtraight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram BKLC. and becauſe each of the angles BAK, AKB is given, the triangle
a. 43. Dat. ABK is given [a] in ſpecies; therefore the ratio of AB to BK is given. and becauſe, by the hypotheſis, the ratio of the parallelogram AC

to EG is given, and that AC is equal[b] to BL; therefore the ratio b. 35. 1.
of BL to EG is given. and becauſe BL is equiangular to EG, and
by the hypotheſis, the ratio of BC to FG is given; therefore[c] the c. 65. Dat.
ratio of KB to EF is given. and the ratio of KB to BA is given; the ratio therefore[d] of AB to EF is given. d. 9. Dat.

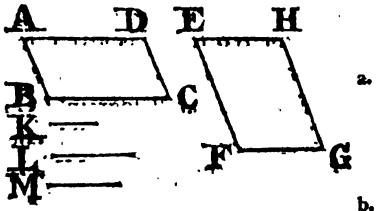


The ratio of AB to EF may be found thus; take the ſtraight line MN given in poſition and magnitude; and make the angle NMO equal to the given angle BAK, and the angle MNO equal to the given angle EFG or AKB. and becauſe the parallelogram BL is equiangular to EG, and has a given ratio to it, and that the ratio of BC to FG is given; find by the 65. Dat. the ratio of KB to EF; and make the ratio of NO to OP the ſame with it. then the ratio of AB to EF is the ſame with the ratio of MO to OP, for ſince the triangle ABK is equiangular to MON, as AB to BK, ſo is MO to ON; and as KB to EF, ſo is NO to OP; therefore, ex aequali, as AB to EF, ſo is MO to OP.

PROP. LXVII. 70.

IF the ſides of two equiangular parallelograms have See N.
given ratios to one another; the parallelograms
ſhall have a given ratio to one another.

Let ABCD, EFGH be two equiangular parallelograms, and let the ratio of AB to EF, as alſo the ratio of BC to FG be given; the ratio of the parallelogram AC to EG is given.



Take a ſtraight line K given in magnitude, and becauſe the ratio
of AB to EF is given, make the
ratio of K to L the ſame with it;
therefore L is given[a], and be- a. 2. Dat.
cauſe the ratio of BC to FG is
given, make the ratio of L to M
the ſame, therefore M is given[a];
and K is given, wherefore[b] the b. 1. Dat.
ratio of K to M is given. but the parallelogram AC is to the parallelogram EG, as the ſtraight line K to the ſtraight line M, as

is demonſtrated in the 23. Prop. of B. 6. Elem. therefore the ratio of AC to EG is given.

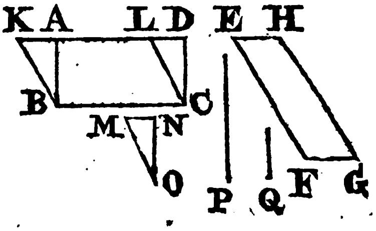
From this it is plain how the ratio of two equiangular parallelograms may be found when the ratios of their ſides are given.

70. PROP. LXVIII.

See N. IF the ſides of two parallelograms which have unequal, but given, angles, have given ratios to one another; the parallelograms ſhall have a given ratio to one another.

Let two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG have the ratios of their ſides, viz. of AB to EF, and of BC to FG given; the ratio of the parallelogram AC to EG is given.

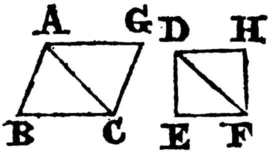
At the point B of the ſtraight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram KBCL. and becauſe each of the angles BAK, BKA is given, the triangle a. 43. Dat. ABK is given [a] in ſpecies. therefore the ratio of AB to BK is b. 9. Dat. given; and the ratio of AB to EF is given, wherefore [b] the ratio of BK to EF is given. and the ratio of BC to FG is given; and the angle KBC is equal c. 67. Dat. to the angle EFG; therefore [c] the ratio of the parallelogram KC to EG is given. but KC is d. 35. 1. equal [d] to AC; therefore the ratio of AC to EG is given.



The ratio of the parallelogram AC to EG may be found thus; take the ſtraight line MN given in poſition and magnitude, and make the angle MNO equal to the given angle KAB, and the angle NMO equal to the given angle AKB or FEH. and becauſe the ratio of AB to EF is given, make the ratio of NO to P the ſame; alſo make the ratio of P to Q the ſame with the given ratio of BC to FG. the parallelogram AC is to EG, as MO to Q.

Becauſe the angle KAB is equal to the angle MNO, and the angle AKB equal to the angle NMO; the triangle AKB is equiangular to NMO. therefore as KB to BA, ſo is MO to ON; and as BA to EF, ſo is NO to P; wherefore, ex aequali, as KB to EF, ſo is MO to P. and BC is to FG, as P to Q; and the parallelograms KC, EG are equiangular; therefore, as was ſhewn in Prop. 67. the parallelogram KC, that is AC, is to EG, as MO to Q.

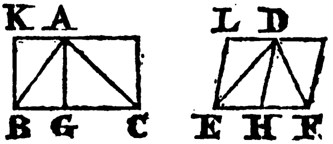
Cor. 1. If two triangles ABC, DEF have two equal angles, 71. or two unequal, but given angles ABC, DEF, and if the ratios of the ſides about theſe angles, viz. the ratios of AB to DE, and of BC to EF be given; the triangles ſhall have a given ratio to one another.



Complete the parallelograms BG, EH; the ratio of BG to EH is given[a]; and therefore the triangles which are the halves[b] of them have a given[c] ratio to one another.

a. 67. or 68. Dat.
b. 34. 1.
c. 15. 5.

Cor. 2. If the baſes BC, EF of two triangles ABC, DEF have 72. a given ratio to one another, and if alſo the ſtraight lines AG, DH which are drawn to the baſes from the oppoſite angles, either in equal angles, or unequal, but given, angles AGC, DHF have a given ratio to one another; the triangles ſhall have a given ratio to one another.



Draw BK, EL parallel to AG, DH, and complete the parallelograms KC, LF. and becauſe the angles AGC, DHF, or their equals, the angles KBC, LEF are either equal, or unequal, but given; and that the ratio of AG to DH, that is of KB to LE is given, as alſo the ratio of BC to EF; therefore[a] the ratio of the parallelogram KC to LF is given. wherefore alſo the ratio of the triangle ABC to DEF is given[b].

a. 67. or 68. Dat.
b. 41. 1. 15. 5.

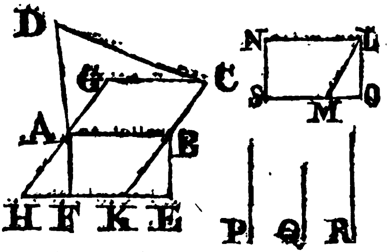
PROP. LXIX. 61.

IF a parallelogram which has a given angle be applied to one ſide of a rectilineal figure given in ſpecies; if the figure have a given ratio to the parallelogram, the parallelogram is given in ſpecies.

Let ABCD be a rectilineal figure given in ſpecies, and to one ſide of it AB let the parallelogram ABEF having the given angle ABE be applied; if the figure ABCD has a given ratio to the parallelogram BF, the parallelogram BF is given in ſpecies.

Thro' the point A draw AG parallel to BC, and thro' the point C draw CG parallel to AB, and produce GA, CB to the points H,

a. 3. Def. K. becauſe the angle ABC is given [a], and the ratio of AB to BC
is given, the figure ABCD being given in ſpecies; therefore the parallelogram BG is given [a] in ſpecies. and becauſe upon the ſame
ſtraight line AB the two rectilineal figures BD, BG given in ſpecies
b. 53. Dat. are deſcribed, the ratio of BD to BG is given [b]. and, by hypothe-
c. 9. Dat. ſis, the ratio of BD to the parallelogram BF is given; wherefore [c]
d. 35. 1. the ratio of BF, that is [d] of the parallelogram BH, to BG is given,
e. 1. 6. and therefore [e] the ratio of the ſtraight line KB to BC is given. and
the ratio of BC to BA is given, wherefore the ratio of KB to BA is given [c]. and becauſe the angle ABC is given, the adjacent angle ABK is given; and the angle ABE is given, therefore the remaining angle KBE is given. the angle EKB is alſo given, becauſe it is equal to the angle ABK; therefore the triangle BKE is given in ſpecies, and conſequently the ratio of EB to BK is given. and the ratio of KB to BA is given, wherefore [c] the ratio of EB to BA is given. and the angle ABE is given, therefore the parallelogram BF is given [a] in ſpecies.



A parallelogram ſimilar to BF may be found thus; take a ſtraight line LM given in poſition and magnitude; and becauſe the angles ABK, ABE are given, make the angle NLM equal to ABK, and the angle NLO equal to ABE. and becauſe the ratio of BF to BD is given, make the ratio of LM to P the ſame with it; and becauſe the ratio of the figure BD to BG is given, find this ratio by the 53. Dat. and make the ratio of P to Q the ſame. alſo, becauſe the ratio of CB to BA is given, make the ratio of Q to R the ſame. and take LN equal to R, thro' the point M draw OM parallel to LN, and complete the parallelogram NLOS; then this is ſimilar to the parallelogram BF.

Becauſe the angle ABK is equal to NLM, and the angle ABE to NLO; the angle KBE is equal to MLO. and the angles BKE, LMO are equal, becauſe the angle ABK is equal to NLM. therefore the triangles BKE, LMO are equiangular to one another, wherefore as BE to BK, ſo is LO to LM. and becauſe as the figure BF to BD, ſo is the ſtraight line LM to P, and as BD to BG, ſo is P to Q; ex aequali, as BF, that is [d] BH, to BG, ſo is LM to Q. but BH is to [e] BG, as KB to BC; as therefore KB to BC, ſo is

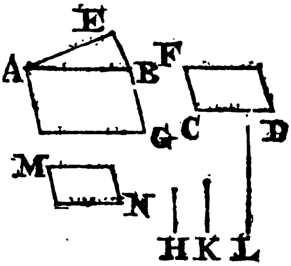
LM to Q. and becauſe BE is to BK, as LO to LM; and as BK to BC, ſo is LM to Q; and as BC to BA, ſo Q was made to R; therefore, ex æquali, as BE to BA, ſo is LO to R; that is to LN. and the angles ABE, NLO are equal; therefore the parallelogram BF is ſimilar to LS.

PROP. LXX. 62. 78.

IF two ſtraight lines have a given ratio to one another, See N. and upon one of them be deſcribed a rectilineal figure given in ſpecies, and upon the other a parallelogram having a given angle; if the figure have a given ratio to the parallelogram, the parallelogram is given in ſpecies.

Let the two ſtraight lines AB, CD have a given ratio to one another, and upon AB let the figure AEB given in ſpecies be deſcribed, and upon CD the parallelogram DF having the given angle FCD; if the ratio of AEB to DF be given, the parallelogram DF is given in ſpecies.

Upon the ſtraight line AB conceive the parallelogram AG to be deſcribed ſimilar and ſimilarly placed to FD. and becauſe the ratio of AB to CD is given, and upon them are deſcribed the ſimilar rectilineal figures AG, FD; the ratio of AG to FD is given [a]. a. 54. Dat. and the ratio of FD to AEB is given; therefore [b] the b. 9. Dat. ratio of AEB to AG is given; and the angle ABG is given, becauſe it is equal to the angle FCD. becauſe therefore the parallelogram AG which has a given angle ABG is applied to a ſide AB of the figure AEB given in ſpecies, and the ratio of AEB to AG is given, the parallelogram AG is given [c] in c. 69. Dat. ſpecies. but FD is ſimilar to AG; therefore FD is given in ſpecies.



A parallelogram ſimilar to FD may be found thus; take a ſtraight line H given in magnitude; and becauſe the ratio of the figure AEB to FD is given, make the ratio of H to K the ſame with it. alſo becauſe the ratio of the ſtraight line CD to AB is given, find by the 54. Dat. the ratio which the figure FD deſcribed upon CD has to the figure AG deſcribed upon AB ſimilar to FD; and make the ratio of K to L the ſame with this ratio. and becauſe the ratios of H to K, and of K to L are given, the ratio of H to L is given [b].

becauſe therefore as AEB to FD, ſo is H to K; and as FD to AG, ſo is K to L; ex aequali, as AEB to AG, ſo is H to L; therefore the ratio of AEB to AG is given. and the figure AEB is given in ſpecies, and to its ſide AB the parallelogram AG is applied in the given angle ABG, therefore by the 69. Dat. a parallelogram may be found ſimilar to AG. let this be the parallelogram MN; MN alſo is ſimilar to FD. for, by the conſtruction, MN is ſimilar to AG, and AG is ſimilar to FD; therefore the parallelogram FD is ſimilar to MN.

81. PROP. LXXI.

IF the extremes of three proportional ſtraight lines have given ratios to the extremes of other three proportional ſtraight lines; the means ſhall alſo have a given ratio to one another. and if one extreme has a given ratio to one extreme, and the mean to the mean; likewiſe the other extreme ſhall have to the other a given ratio.

Let A, B, C be three proportional ſtraight lines, and D, E, F three other; and let the ratios of A to D, and of C to F be given. then the ratio of B to E is alſo given.

Becauſe the ratio of A to D, as alſo of C to F is given, the
a. 67. Dat. ratio of the rectangle A, C to the rectangle D, F is given [a]. but
b. 17. 6. the ſquare of B is equal [b] to the rectangle A, C; and the ſquare of E to the rectangle [b] D, F. therefore the ratio of the ſquare
c. 58. Dat. of B to the ſquare of E is given; wherefore [c] alſo the ratio of the ſtraight line B to E is given.

Next, let the ratio of A to D, and of B to E be given; then the ratio of C to F is alſo given.

Becauſe the ratio of B to E is given, the ratio of the
d. 54. Dat. ſquare of B to the ſquare of E is given [d]. therefore [b] the ratio of the rectangle A, C to the rectangle D, F is given. and the ratio of the ſide A to the ſide D is given; there-
e. 65. Dat. fore the ratio of the other ſide C to the other F is given [e].

Cor. And if the extremes of four proportionals have to the extremes of four other proportionals given ratios, and one of the means a given ratio to one of the means; the other mean ſhall have a given ratio to the other mean. as may be ſhewn in the ſame manner as in the foregoing Propoſition.

PROP. LXXII.

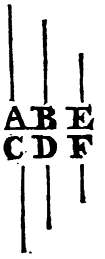
82.

IF four ſtraight lines be proportionals; as the firſt is to the ſtraight line to which the ſecond has a given ratio; ſo is the third to the ſtraight line to which the fourth has a given ratio.

Let A, B, C, D be four proportional ſtraight lines, viz. as A to B, ſo C to D; as A is to the ſtraight line to which B has a given ratio, ſo is C to the ſtraight line to which D has a given ratio.

Let E be the ſtraight line to which B has a given ratio, and as B to E, ſo make D to F. the ratio of B to E is given [a], and therefore the ratio of D to F. and becauſe as A to B, ſo is C to D; and as B to E, ſo D to F; therefore, ex aequali, as A to E, ſo is C to F. and E is the ſtraight line to which B has a given ratio, and F that to which D has a given ratio; therefore as A is to the ſtraight line to which B has a given ratio, ſo is C to that to which D has a given ratio.

a. Hyp.



PROP. LXXIII.

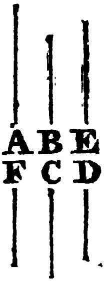
83.

IF four ſtraight lines be proportionals; as the firſt is to the ſtraight line to which the ſecond has a given ratio, ſo is the ſtraight line to which the third has a given ratio to the fourth. See N.

Let the ſtraight line A be to B, as C to D; as A to the ſtraight line to which B has a given ratio, ſo is the ſtraight line to which C has a given ratio to D.

Let E be the ſtraight line to which B has a given ratio, and as B to E, ſo make F to C; becauſe the ratio of B to E is given, the ratio of C to F is given. and becauſe A is to B, as C to D; and as B to E, ſo F to C; therefore, ex aequali in proportione perturbata [a], A is to E, as F to D; that is A is to E to which B has a given ratio, as F, to which C has a given ratio, is to D.

a. 23. 5.

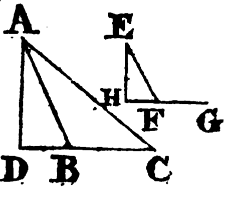


64. PROP. LXXIV.

IF a triangle has a given obtuſe angle; the exceſs of the ſquare of the ſide which ſubtends the obtuſe angle above the ſquares of the ſides which contain it, ſhall have a given ratio to the triangle.

Let the triangle ABC have a given obtuſe angle ABC; and produce the ſtraight line CB, and from the point A draw AD perpendicular to BC. the exceſs of the ſquare of AC above the
a. 12. 2. ſquares of AB, BC, that is [a] the double of the rectangle contained by DB, BC, has a given ratio to the triangle ABC.

Becauſe the angle ABC is given, the angle ABD is alſo given;
b. 43. Dat. and the angle ADB is given, wherefore the triangle ABD is given[b] in ſpecies; and therefore the ratio of AD to DB is given. and as
c. 1. 6. AD to DB, ſo is [c] the rectangle AD, BC to the rectangle DB, BC; wherefore the ratio of the rectangle AD, BC to the rectangle DB, BC is given, as alſo the ratio of twice the rectangle DB, BC to the rectangle AD, BC. but the ratio of the rectangle AD, BC to the triangle ABC is given,
d. 41. 1. becauſe it is double [d] of the triangle; therefore the ratio of twice the rectangle DB, BC
e. 9. Dat. to the triangle ABC is given [e]. and twice the rectangle DB, BC is the exceſs [a] of the ſquare of AC above the ſquares of AB, BC. therefore this exceſs has a given ratio to the triangle ABC.



And the ratio of this exceſs to the triangle ABC may be found thus; take a ſtraight line EF given in poſition and magnitude; and becauſe the angle ABC is given, at the point F of the ſtraight line EF make the angle EFG equal to the angle ABC; produce GF, and draw EH perpendicular to FG. then the ratio of the exceſs of the ſquare of AC above the ſquares of AB, BC to the triangle ABC is the ſame with the ratio of quadruple the ſtraight line HF to HE.

Becauſe the angle ABD is equal to the angle EFH, and the angle ADB to EHF, each being a right angle; the triangle ADB is
f. 4. 6. equiangular to EHF. therefore [f] as BD to DA, ſo FH to HE; and
g. Cor. 4. 5. as quadruple of BD to DA, ſo is [g] quadruple of FH to HE. but as twice BD is to DA, ſo is [c] twice the rectangle DB, BC to the
h. C. 5. rectangle AD, BC; and as DA to the half of it, ſo is [h] the rectangle AD, BC to its half the triangle ABC; therefore, ex ae-

quali, as twice BD is to the half of DA, that is, as quadruple of BD is to DA, that is, as quadruple of FH to HE, ſo is twice the rectangle DB, BC to the triangle ABC.

PROP. LXXV. 65.

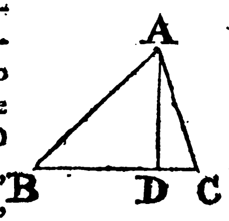
IF a triangle has a given acute angle; the ſpace by which the ſquare of the ſide ſubtending the acute angle is leſs than the ſquares of the ſides which contain it, ſhall have a given ratio to the triangle.

Let the triangle ABC have a given acute angle ABC, and draw AD perpendicular to BC; the ſpace by which the ſquare of AC is leſs than the ſquares of AB, BC, that is [a] the double of the rectangle contained by CB, BD, has a given ratio to the triangle ABC.

a. 13. 2.

Becauſe the angles ABD, ADB are each of them given, the triangle ABD is given in ſpecies; and therefore the ratio of BD to DA is given. and as BD to DA, ſo is the rectangle CB, BD to the rectangle CB, AD; therefore the ratio of theſe rectangles is given, as alſo the ratio of twice the rectangle CB, BD to the rectangle CB, AD. but the rectangle CB, AD has a given ratio to its half the triangle ABC, therefore [b] the ratio of twice the rectangle CB, BD to the triangle ABC is given. and twice the rectangle CB, BD is [a] the ſpace by which the ſquare of AC is leſs than the ſquares of AB, BC; therefore the ratio of this ſpace to the triangle ABC is given. and the ratio may be found as in the preceding Propoſition.

b. 9. Dat.

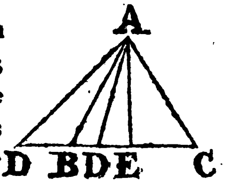


LEMMA.

IF from the vertex A of an Iſoſceles triangle ABC, any ſtraight line AD be drawn to the baſe BC; the ſquare of the ſide AB is equal to the rectangle BD, DC of the ſegments of the baſe together with the ſquare of AD. but if AD be drawn to the baſe produced, the ſquare of AD is equal to the rectangle BD, DC together with the ſquare of AB.

Cas. 1. Biſect the baſe BC in E, and join AE which will be perpendicular [a] to BC; wherefore the ſquare of AB is equal [b] to the ſquares of AE, EB. but the ſquare of EB is equal [c] to the rectangle BD, DC together with the ſquare of DE. therefore the ſquare

a. 8. 1.
b. 47. 1.
c. 5. 2.



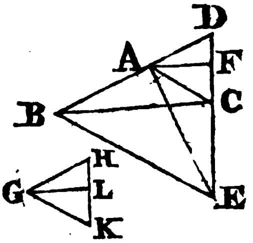
b. 47. 1. of AB is equal to the ſquares of AE, ED, that is to [b] the ſquare of AD, together with the rectangle BD, DC. the other caſe is ſhewn in the ſame way by 6. 2. Elem.

67. PROP. LXXVI.

IF a triangle have a given angle, the exceſs of the ſquare of the ſtraight line which is equal to the two ſides that contain the given angle, above the ſquare of the third ſide, ſhall have a given ratio to the triangle.

Let the triangle ABC have the given angle BAC, the exceſs of the ſquare of the ſtraight line which is equal to BA, AC together above the ſquare of BC, ſhall have a given ratio to the triangle ABC.

Produce BA, and take AD equal to AC, join DC and produce it to E, and thro' the point B draw BE parallel to AC; join AE, and draw AF perpendicular to DC. and becauſe AD is equal to AC, BD is equal to BE; and BC is drawn from the vertex B of the Iſoſceles triangle DBE, therefore, by the Lemma, the ſquare of BD, that is of BA and AC together, is equal to the rectangle DC, CE together with the ſquare of BC; and therefore the ſquare of BA, AC together, that is of BD, is greater than the ſquare of BC by the rectangle DC, CE; and this rectangle has a given ratio to the triangle ABC. becauſe the angle BAC is given, the adjacent angle CAD is given; and each of the angles ADC, DCA is given, for each of them is
a. 5. & 32. 1. the half [a] of the given angle BAC; there-
b. 43. Dat. fore the triangle ADC is given [b] in ſpecies; and AF is drawn from its vertex to the baſe in a given angle,
c. 50. Dat. wherefore the ratio of AF to the baſe CD is given [c]. and as CD
d. 1. 6. to AF, ſo is [d] the rectangle DC, CE to the rectangle AF, CE; and
e. 41. 1. the ratio of the rectangle AF, CE to its half [e] the triangle ACE is given; therefore the ratio of the rectangle DC, CE to the triangle
f. 37. 1. ACE, that is [f] to the triangle ABC is given [g]. and the rectangle DC,
g. 9. Dat. CE is the exceſs of the ſquare of BA, AC together above the ſquare of BC; therefore the ratio of this exceſs to the triangle ABC is given.



The ratio which the rectangle DC, CE has to the triangle ABC is found thus. take the ſtraight line GH given in poſition and magnitude, and at the point G in GH make the angle HGK equal to the given angle CAD, and take GK equal to GH, join KH, and draw GL perpendicular to it. then the ratio of HK to the half of GL is the ſame with the ratio of the rectangle DC, CE to the triangle ABC. becauſe the angles HGK, DAC at the vertices of the Iſoſceles triangles GHK, ADC are equal to one another, theſe triangles are ſimilar, and becauſe GL, AF are perpendicular to the baſes HK, DC, as HK to GL, ſo is [h] (DC to AF, and ſo is) the rectangle DC, CE to the rectangle AF, CE; but as GL to its half, ſo is the rectangle AF, CE to its half which is the triangle ACE, or the triangle ABC; therefore, ex aequali, HK is to the half of the ſtraight line GL, as the rectangle DC, CE is to the triangle ABC. h. {4. 6. 22. 5.

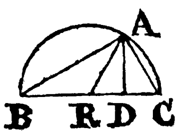
Cor. And if a triangle have a given angle, the ſpace by which the ſquare of the ſtraight line which is the difference of the ſides which contain the given angle is leſs than the ſquare of the third ſide, ſhall have a given ratio to the triangle. this is demonſtrated the ſame way as the preceding Propoſition, by help of the ſecond caſe of the Lemma.

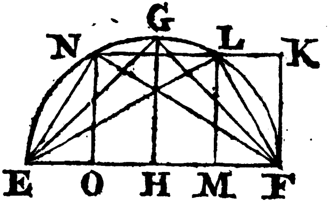
PROP. LXXVII. I.

IF the perpendicular drawn from a given angle of a triangle to the oppoſite ſide, or baſe, has a given ratio to the baſe; the triangle is given in ſpecies. See N.

Let the triangle ABC have the given angle BAC, and let the perpendicular AD drawn to the baſe BC, have a given ratio to it; the triangle ABC is given in ſpecies.

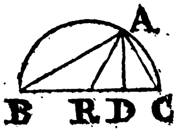
If ABC be an Iſoſceles triangle, it is evident [a] that if any one of its angles be given, the reſt are alſo given; and therefore the triangle is given in ſpecies, without the conſideration of the ratio of the perpendicular to the baſe, which in this caſe is given by Prop. 50. a. 5. & 32. 1.

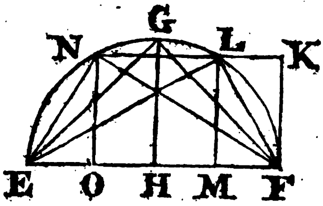




But when ABC is not an Iſoſceles triangle, take any ſtraight line EF given in poſition and magnitude, and upon it deſcribe the ſeg-

ment of a circle EGF containing an angle equal to the given angle
BAC; draw GH biſecting EF at right angles, and join EG, GF.
then ſince the angle EGF is equal to the angle BAC, and that
EGF is an Iſoſceles triangle and ABC is not, the angle FEG is not
equal to the angle CBA. draw EL making the angle FEL equal to
the angle CBA, join FL, and draw LM perpendicular to EF. then
becauſe the triangles ELF, BAC are equiangular, as alſo are the triangles
MLE, DAB, as ML to LE, ſo is DA to AB; and as LE to
EF, ſo is AB to BC; wherefore, ex aequali, as LM to EF, ſo is
AD to BC, and becauſe the ratio of AD to BC is given, therefore
b. 2. Dat. the ratio of LM to EF is given; and EF is given, wherefore [b] LM
alſo is given. complete the parallelogram LMFK, and becauſe LM
is given, FK is given in magnitude; it is alſo given in poſition,
c. 30. Dat. and the point F is given, and conſequently [c] the point K; and becauſe
thro' K the ſtraight line KL is drawn parallel to EF which
d. 31. Dat. is given in poſition, therefore [d] KL is given in poſition; and the
circumference ELF is given in poſition, therefore the point L is
e. 28. Dat. given [e]. and becauſe the points L, E, F are given, the ſtraight lines
f. 29. Dat. LE, EF, FL are given [f] in magnitude; therefore, the triangle LEF
g. 42. Dat. is given in ſpecies [g]. and the triangle ABC is ſimilar to LEF,
wherefore alſo ABC is given in ſpecies.





Becauſe LM is leſs than GH, the ratio of LM to EF, that is the given ratio of AD to BC muſt be leſs than the ratio of GH to EF which the ſtraight line, in a ſegment of a circle containing an angle equal to the given angle, that biſects the baſe of the ſegment at right angles, has unto the baſe.

Cor. 1. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if the perpendicular AD be to the baſe BC, as the perpendicular LM to the baſe EF; the triangles ABC, LEF are ſimilar.

Deſcribe the circle EGF about the triangle ELF, and draw LN parallel to EF, join EN, NF, and draw NO perpendicular to EF. becauſe the angles ENF, ELF are equal, and that the angle EFN is equal to the alternate angle FNL, that is to the angle FEL in the ſame ſegment, therefore the triangle NEF is ſimilar to LEF, and in the ſegment EGF there can be no other triangle upon the baſe EF which has the ratio of its perpendicular to that baſe the ſame with the ratio of LM or NO to EF, becauſe the perpendicular muſt be greater or leſs than LM or NO. but, as has been ſhewn in the preceding demonſtration, a triangle ſimilar to ABC can be deſcribed in the ſegment EGF upon the baſe EF, and the ratio of its perpendicular to the baſe is the ſame, as was there ſhewn, with the ratio of AD to BC, that is of LM to EF. therefore that triangle muſt be either LEF, or NEF, which therefore are ſimilar to the triangle ABC.

Cor. 2. If a triangle ABC has a given angle BAC, and if the ſtraight line AR drawn from the given angle to the oppoſite ſide BC, in a given angle ARC, has a given ratio to BC; the triangle ABC is given in ſpecies.

Draw AD perpendicular to BC; therefore the triangle ARD is
given in ſpecies; wherefore the ratio of AD to AR is given; and the
ratio of AR to BC is given, and conſequently [h] the ratio of AD h. 9. Dat.
to BC is given; and the triangle ABC is therefore given in ſpecies [i]. i. 77. Dat.

Cor. 3. If two triangles ABC, LEF have one angle BAC
equal to one angle ELF, and if ſtraight lines drawn from theſe
angles to the baſes, making with them given and equal angles, have
the ſame ratio to the baſes, each to each; then the triangles are
ſimilar. for, having drawn perpendiculars to the baſes from the
equal angles, as one perpendicular is to its baſe, ſo is the other to
its baſe [k]. wherefore, by Cor. 1. the triangles are ſimilar. k. 4. 6. 22. 5.

A triangle ſimilar to ABC may be found thus; having deſcribed the ſegment EGF and drawn the ſtraight line GH as was directed in the Propoſition, find FK which has to EF the given ratio of AD to BC; and place FK at right angles to EF from the point F. then becauſe, as has been ſhewn, the ratio of AD to BC, that is of FK to EF, muſt be leſs than the ratio of GH to EF; therefore FK is leſs than GH; and conſequently the parallel to EF drawn through the point K muſt meet the circumference of the ſegment in two points. let L be either of them, and join EL, LF, and draw LM perpendicular to EF. then becauſe the angle BAC is equal to the angle ELF, and that AD is to BC, as KF, that is LM to EF, the triangle ABC is ſimilar to triangle LEF. by Cor. 1.

80.

PROP. LXXVIII.

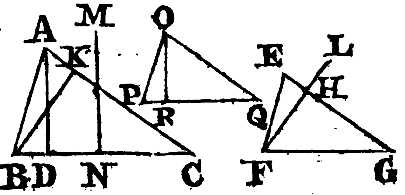
IF a triangle have one angle given, and if the ratio of the rectangle of the ſides which contain the given angle to the ſquare of the third ſide be given; the triangle is given in ſpecies.

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the ſquare of BC be given; the triangle ABC is given in ſpecies.

From the point A draw AD perpendicular to BC; the rectangle AD, BC has a given ratio to its half [a] the triangle ABC. and becauſe the angle BAC is given, the ratio of the triangle ABC to the rectangle BA, AC is given [b]; and, by the hypotheſis, the ratio of the rectangle BA, AC to the ſquare of BC is given. therefore [c] the ratio of the rectangle AD, BC to the ſquare of BC, that is [d] the ratio of the ſtraight line AD to BC is given. wherefore the triangle ABC is given [e] in ſpecies.

a. 41. 1.
b. Cor. 62. Dat.
c. 9. Dat.
d. 1. 6.
e. 77. Dat.

A triangle ſimilar to ABC may be found thus; take a ſtraight line EF given in poſition and magnitude, and make the angle FEG equal to the given angle BAC, and draw FH perpendicular to EG, and BK perpendicular to AC; therefore the triangles ABK, EFH are ſimilar. and the rectangle AD, BC, or the rectangle BK, AC, which is equal to it, is to the rectangle BA, AC, as the ſtraight line BK to BA, that is as FH to FE; let the given ratio of the rectangle BA, AC to the ſquare of BC be the ſame with the ratio of the ſtraight line EF to FL; therefore, ex aequali, the ratio of the rectangle AD, BC to the ſquare of BC, that is the ratio of the ſtraight line AD to BC, is the ſame with the ratio of HF to FL. and becauſe AD is not greater than the ſtraight line MN in the ſegment of the circle deſcribed about the triangle ABC, which biſects BC at right angles; the ratio of AD to BC, that is of HF to FL, muſt not be greater than the ratio of MN to BC. let it be ſo, and by the 77. Dat. find a triangle OPQ which has one of its angles POQ equal to the given angle BAC, and the ratio of the perpendicular OR, drawn from that angle, to the baſe PQ the ſame with the ratio of HF to FL. then the triangle ABC is ſimilar to



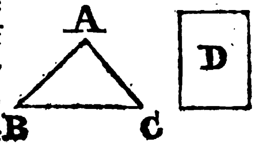
OPQ. becauſe, as has been ſhewn, the ratio of AD to BC is the ſame with the ratio of (HF to FL, that is, by the conſtruction, with the ratio of) OR to PQ; and the angle BAC is equal to the angle POQ. therefore the triangle ABC is ſimilar [f] to the triangle POQ.

f. 1. Cor. 77. Dat.

Otherwiſe,

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the ſquare of BC be given; the triangle ABC is given in ſpecies.

Becauſe the angle BAC is given, the exceſs of the ſquare of both the ſides BA, AC together above the ſquare of the third ſide BC has a given [a] ratio to the triangle ABC. let the figure D be equal to this exceſs; therefore the ratio of D to the triangle ABC is given; and the ratio of the triangle ABC to the rectangle BA, AC is given [b], becauſe BAC is a given angle; and the rectangle BA, AC has a given ratio to the ſquare of BC; wherefore [c] the ratio of D to the ſquare of BC is given. and, by compoſition [d], the ratio of the ſpace D together with the ſquare of BC to the ſquare of BC is given. but D together with the ſquare of BC is equal to the ſquare of both BA and AC together; therefore the ratio of the ſquare of BA, AC together to the ſquare of BC is given; and the ratio of BA, AC together to BC is therefore given [e]. and the angle BAC is given, wherefore [f] the triangle ABC is given in ſpecies.



a. 76. Dat.
b. Cor. 62. Dat.
c. 10. Dat.
d. 7. Dat.
e. 59. Dat.
f. 48. Dat.

The compoſition of this which depends upon thoſe of the 76. and 48. Propoſitions is more complex than the preceding compoſition which depends upon that of Prop. 77. which is eaſy.

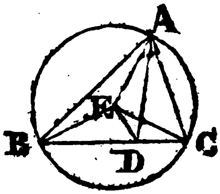
PROP. LXXIX. K.

See N.

IF a triangle have a given angle, and if the ſtraight line drawn from that angle to the baſe, making a given angle with it, divides the baſe into ſegments which have a given ratio to one another; the triangle is given in ſpecies.

Let the triangle ABC have the given angle BAC, and let the ſtraight line AD drawn to the baſe BC making the given angle ADB, divide BC into the ſegments BD, DC which have a given ratio to one another; the triangle ABC is given in ſpecies.

a. 5. 4. Deſcribe [a] the circle BAC about the triangle, and from its center E draw EA, EB, EC, ED. becauſe the angle BAC is given, the
b. 20. 3. angle BEC at the center, which is the double [b] of it, is given. and the ratio of BE to EC is given, becauſe they are equal to one ano-
c. 44. Dat. ther; therefore [c] the triangle BEC is given in ſpecies, and the ratio
d. 7. Dat. of EB to BC is given. alſo the ratio of CB to BD is given [d], becauſe the ratio of BD to DC is given; therefore the ratio of EB to
e. 9. Dat. BD is given [e]. and the angle EBC is given, wherefore the triangle EBD is given [c] in ſpecies, and the ratio of EB, that is of EA to ED is therefore given. and the angle EDA is given, becauſe each of the
f. 47. Dat. angles BDE, BDA is given. therefore the triangle AED is given [f] in ſpecies, and the angle AED given; alſo the angle DEC is given, becauſe each of the angles BED, BEC is given; therefore the angle AEC is given. and the ratio of EA to EC, which are equal, is given; and the triangle AEC is therefore given [c] in ſpecies, and the angle ECA given. and the angle ECB is given, wherefore the angle ACB is given. and the angle
g. 43. Dat. BAC is alſo given; therefore [g] the triangle ABC is given in ſpecies.



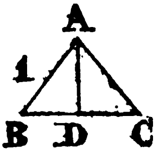
A triangle ſimilar to ABC may be found, by taking a ſtraight line given in poſition and magnitude, and dividing it in the given ratio which the ſegments BD, DC are required to have to one another; then if upon that ſtraight line a ſegment of a circle be deſcribed containing an angle equal to the given angle BAC, and a ſtraight line be drawn from the point of diviſion in an angle equal to the given angle ADB, and from the point where it meets the circumference, ſtraight lines be drawn to the extremity of the firſt line, theſe together with the firſt line ſhall contain a triangle ſimilar to ABC, as may eaſily be ſhewn.

The Demonſtration may be alſo made in the manner of that of the 77. Prop. and that of the 77. may be made in the manner of this.

L. PROP. LXXX.

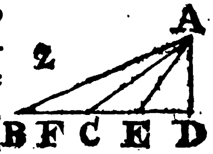
IF the ſides about an angle of a triangle have a given ratio to one another, and if the perpendicular drawn from that angle to the baſe has a given ratio to the baſe; the triangle is given in ſpecies.

Let the ſides BA, AC, about the angle BAC of the triangle ABC have a given ratio to one another, and let the perpendicular AD have a given ratio to the baſe BC; the triangle ABC is given in ſpecies.

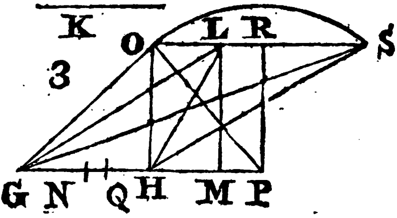


Firſt, let the ſides AB, AC be equal to one another, therefore the perpendicular AD biſects [a] the baſe BC. and a. 26. 1. the ratio of AD to BC, and therefore to its half DB is given; and the angle ADB is given. wherefore the triangle [*] ABD and conſequently the tri- * 43. Dat. angle ABC is given [b] in ſpecies. b. 44. Dat.

But let the ſides be unequal, and BA be greater than AC; and make the angle CAE equal to the angle ABC. becauſe the angle AEB is common to the triangles AEB, CEA, they are ſimilar; therefore as AB to BE, ſo is CA to AE, and, by permutation, as BA to AC, ſo is BE to EA, and ſo is EA to EC. and the ratio of BA to AC is given, therefore the ratio of BE to EA, and the ratio of EA to EC, as alſo the ratio of BE to EC is given [c]; wherefore c. 9. Dat. the ratio of EB to BC is given [d]. and the ratio d. 6. Dat. of AD to BC is given by the Hypotheſis, therefore [c] the ratio of AD to BE is given; and the ratio of BE to EA was ſhewn to be given; wherefore the ratio of AD to AE is given. and ADE is a right angle, therefore the triangle ADE is given [e] in ſpe- e. 46. Dat. cies, and the angle AEB given; the ratio of BE to EA is likewiſe given, therefore [b] the triangle ABE is given in ſpecies, and conſequently the angle EAB, as alſo the angle ABE, that is the angle CAE is given; therefore the angle BAC is given, and the angle ABC being alſo given, the triangle ABC is given [f] in ſpecies. f. 43. Dat.



How to find a triangle which ſhall have the things which are mentioned to be given in the Propoſition, is evident in the firſt caſe. and to find it the more eaſily in the other caſe, it is to be obſerved that if the ſtraight line EF equal to EA be placed in EB towards B, the point F divides the baſe BC into the ſegments BF, FC which have to one another the ratio of the ſides BA, AC. becauſe BE, EA, or EF, and EC were ſhewn to be proportionals, therefore [*] BF is * 19. 5. to FC, as BE to EF, or EA, that is as BA to AC. and AE cannot be leſs than the altitude of the triangle ABC, but it may be equal to it; which if it be, the triangle, in this caſe, as alſo the ratio of the ſides, may be thus found, having given the ratio of the perpendicular to the baſe. take the ſtraight line GH given in poſition and magnitude, for the baſe of the triangle to be found; and let the given ratio of the perpendicular to the baſe be that of the ſtraight line K to GH, that is, let K be equal to the perpendicular; and ſuppoſe GLH to be the triangle which is to be found. therefore having made the angle HLM equal to LGH, it is required that LM be perpendicular to GM and equal to K. and becauſe GM, ML, MH are proportionals, as was ſhewn of BE, EA, EC, the rectangle GMH is equal to the ſquare of ML. add the common ſquare of NH, g. 6. 2. (having biſected GH in N) and the ſquare of NM is equal [g] to the ſquares of the given ſtraight lines NH and ML, or K. therefore the ſquare of NM, and its ſide NM, is given, as alſo the point M, viz. by taking the ſtraight line NM the ſquare of which is equal to the ſquares of NH, ML. draw ML equal to K, at right angles to GM. and becauſe ML is given in poſition and magnitude, therefore the point L is given; join LG, LH, then the triangle LGH is that which was to be found. for the ſquare of NM is equal to the ſquares of NH and ML, and taking away the common ſquare of NH, the rectangle GMH is equal [g] to the ſquare of ML; therefore as GM to ML, ſo is ML to MH, and h. 6. 6. the triangle LGM is [h] therefore equiangular to HLM, and the angle HLM equal to the angle LGM, and the ſtraight line LM, drawn from the vertex of the triangle making the angle HLM equal to LGH, is perpendicular to the baſe and equal to the given ſtraight line K, as was required. and the ratio of the ſides GL, LH is the ſame with the ratio of GM to ML, that is with the ratio of the ſtraight line which is made up of GN the half of the given baſe and of NM the ſquare of which is equal to the ſquares of GN and K, to the ſtraight line K.



And whether this ratio of GM to ML is greater or leſs than the ratio of the ſides of any other triangle upon the baſe GH, and of which the altitude is equal to the ſtraight line K, that is, the vertex of which is in the parallel to GH drawn thro' the point L, may be thus found. Let OGH be any ſuch triangle, and draw OP making the angle HOP equal to the angle OGH; therefore, as before, GP, PO, PH are proportionals. and PO cannot be equal to LM, becauſe the rectangle GPH, would be equal to the rectangle GMH, which is impoſſible, for the point P cannot fall upon M, becauſe O would then fall on L; nor can PO be leſs than LM, therefore it is greater; and conſequently the rectangle GPH is greater than the rectangle GMH, and the ſtraight line GP greater than GM. therefore the ratio of GM to MH is greater than the ratio of GP to PH, and the ratio of the ſquare of GM to the ſquare of ML is therefore [i] greater i. 2. Cor. 20. 6. than the ratio of the ſquare of GP to the ſquare of PO, and the ratio of the ſtraight line GM to ML, greater than the ratio of GP to PO. but as GM to ML, ſo is GL to LH; and as GP to PO, ſo is GO to OH; therefore the ratio of GL to LH is greater than the ratio of GO to OH; wherefore the ratio of GL to LH is the greateſt of all others; and conſequently the given ratio of the greater ſide to the leſs muſt not be greater than this ratio.

But if the ratio of the ſides be not the ſame with this greateſt ratio of GM to ML, it muſt neceſſarily be leſs than it. Let any leſs ratio be given, and the ſame things being ſuppoſed, viz. that GH is the baſe, and K equal to the altitude of the triangle, it may be found as follows. Divide GH in the point Q, ſo that the ratio of GQ to QH may be the ſame with the given ratio of the ſides; and as GQ to QH, ſo make GP to PQ, and ſo will [f] PQ be to PH; where- f. 19. 5. fore the ſquare of GP is to the ſquare of PQ, as [i] the ſtraight line GP to PH. and becauſe GM, ML, MH are proportionals, the ſquare of GM is to the ſquare of ML, as [i] the ſtraight line GM to MH. but the ratio of GQ to QH, that is the ratio of GP to PQ, is leſs than the ratio of GM to ML; and therefore the ratio of the ſquare of GP to the ſquare of PQ is leſs than the ratio of the ſquare of GM to that of ML; and conſequently the ratio of the ſtraight line GP to PH is leſs than the ratio of GM to MH, and, by diviſion, the ratio of GH to HP is leſs than that of GH to HM; wherefore [k] k. 10. 5. the ſtraight line HP is greater than HM, and the rectangle GPH, that is the ſquare of PQ, greater than the rectangle GMH, that is than the ſquare of ML, and the ſtraight line PQ is therefore greater than ML. draw LR parallel to GP, and from P draw PR at right angles to GP. becauſe PQ is greater than ML, or PR, the circle deſcribed from the center P, at the diſtance PQ, muſt neceſſarily cut LR in two points; let theſe be O, S, and join OG, OH; SG, SH; each of the triangles OGH, SGH have the things mentioned to be given in the Propoſition. join OP, SP; and becauſe as GP to PQ, or PO, ſo is PO to PH, the triangle OGP is equiangular to HOP; as, therefore, OG to GP, ſo is HO to OP, and, by permutation, as GO to OH, ſo is GP to PO, or PQ, and ſo is GQ to

QH. therefore the triangle OGH has the ratio of its ſides GO, OH the ſame with the given ratio of GQ to QH; and the perpendicular has to the baſe the given ratio of K to GH, becauſe the perpendicular is equal to LM, or K. the like may be ſhewn in the ſame way of the triangle SGH.

This conſtruction by which the triangle OGH is found, is ſhorter than that which would be deduced from the Demonſtration of the Datum; by reaſon that the baſe GH is given in poſition and magnitude, which was not ſuppoſed in the Demonſtration. the ſame thing is to be obſerved in the next Propoſition.

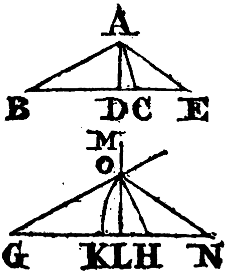
M.

PROP. LXXXI.

IF the ſides about an angle of a triangle be unequal and have a given ratio to one another, and if the perpendicular from that angle to the baſe divides it into ſegments that have a given ratio to one another; the triangle is given in ſpecies.

Let ABC be a triangle the ſides of which about the angle BAC are unequal, and have a given ratio to one another, and let the perpendicular AD to the baſe BC divide it into the ſegments BD, DC which have a given ratio to one another; the triangle ABC is given in ſpecies.

Let AB be greater than AC, and make the angle CAE equal to the angle ABC; and becauſe the angle AEB is common to the triangles ABE, CAE, they are [a] equiangular to one another. therefore as AB to BE, ſo is CA to AE, and, by permutation, as AB to AC, ſo BE to EA, and ſo is EA to EC. but the ratio of BA to AC is given, therefore the ratio of BE to EA, as alſo the ratio of EA to EC is given; wherefore [b] the ratio of BE to EC, as alſo [c] the ratio of EC to CB is given. and the ratio of BC to CD is given [d], becauſe the ratio of BD to DC is given; therefore [b] the ratio of EC to CD is given, and conſequently [d] the ratio of DE to EC. and the ratio of EC to EA was ſhewn to be given, therefore [b] the ratio of DE to EA is given. and ADE is a right angle, wherefore [e] the triangle ADE is given in ſpecies, and the angle AED given. and



a. 4. 6.
b. 9. Dat.
c. Cor. 6. Dat.
d. 7. Dat.
e. 46. Dat.

the ratio of CE to EA is given, therefore [f] the triangle AEC is given in ſpecies, and conſequently the angle ACE is given, as alſo the adjacent angle ACB. in the ſame manner, becauſe the ratio of BE to EA is given, the triangle BEA is given in ſpecies, and the angle ABE is therefore given. and the angle ACB is given; wherefore the triangle ABC is given [g] in ſpecies. f. 44. Dat. g. 43. Dat.

But the ratio of the greater ſide BA to the other AC muſt be leſs than the ratio of the greater ſegment BD to DC. becauſe the ſquare of BA is to the ſquare of AC, as the ſquares of BD, DA to the ſquares of DC, DA; and the ſquares of BD, DA have to the ſquares of DC, DA a leſs ratio than the ſquare of BD has to the ſquare of DC †, becauſe the ſquare of BD is greater than the ſquare of DC; therefore the ſquare of BA has to the ſquare of AC a leſs ratio than the ſquare of BD has to that of DC. and conſequently the ratio of BA to AC is leſs than the ratio of BD to DC.

This being premiſed, a triangle which ſhall have the things mentioned to be given in the Propoſition, and to which the triangle ABC is ſimilar, may be found thus. take a ſtraight line GH given in poſition and magnitude, and divide it in K ſo that the ratio of GK to KH may be the ſame with the given ratio of BA to AC; divide alſo GH in L ſo that the ratio of GL to LH may be the ſame with the given ratio of BD to DC, and draw LM at right angles to GH. and becauſe the ratio of the ſides of a triangle is leſs than the ratio of the ſegments of the baſe, as has been ſhewn, the ratio of GK to KH is leſs than the ratio of GL to LH, wherefore the point L muſt fall betwixt K and H. alſo make as GK to KH, ſo GN to NK, and ſo ſhall [h] NK be to NH. and from the center N, at the diſtance NK deſcribe a circle, and let its circumference meet LM in O, and join OG, OH; then OGH is the triangle which was to be deſcribed. becauſe GN is to NK, or NO, as NO to NH, the triangle OGN is equiangular to HON; therefore as OG to GN, ſo is HO to ON, and, by permutation, as GO to OH, ſo is GN to NO, or NK, that is as GK to KH, that is in the given ratio of the ſides. and, by the conſtruction, GL, LH have to one another the given ratio of the ſegments of the baſe. h. 19. 5.

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† If A be greater than B, and C any third magnitude; then A and C together have to B and C together a leſs ratio than A has to B.

Let A be to B as C to D, and becauſe A is greater than B, C is greater than D. but as A is to B, ſo A and C to B and D; and A and C have to B and C a leſs ratio than A and C have to B and D, becauſe C is greater than D. therefore A and C have to B and C a leſs ratio than A to B.

60. PROP. LXXXII.

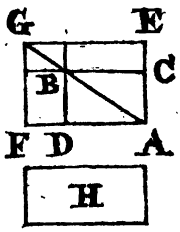
IF a parallelogram given in ſpecies and magnitude be encreaſed, or diminiſhed by a gnomon given in magnitude; the ſides of the gnomon are given in magnitude.

Firſt, let the parallelogram AB given in ſpecies and magnitude be encreaſed by the given gnomon ECBDFG; each of the ſtraight lines CE, DF is given.

Becauſe AB is given in ſpecies and magnitude, and that the gnomon ECBDFG is given, therefore the whole ſpace AG is given in magnitude. but AG is alſo given in ſpecies, becauſe it is ſimilar [a] to AB; therefore the ſides of AG are given [b]. each of the ſtraight lines AE, AF is therefore given; and each of the ſtraight lines CA, AD is given [b], therefore each of the remainders EC, DF is given [c].

a. 1. Def. 2. and 24. 6.
b. 60. Dat.
c. 4. Dat.

Next, let the parallelogram AG given in ſpecies and magnitude be diminiſhed by the given gnomon ECBDFG; each of the ſtraight lines CE, DF is given.



Becauſe the parallelogram AG is given, as alſo its gnomon ECBDFG; the remaining ſpace AB is given in magnitude. but it is alſo given in ſpecies; becauſe it is ſimilar [a] to AG; therefore [b] its ſides CA, AD are given. and each of the ſtraight lines EA, AF is given; therefore EC, DF are each of them given.

The gnomon and its ſides CE, DF may be found thus in the firſt caſe. let H be the given ſpace to which the gnomon muſt be made equal, and find [d] a parallelogram ſimilar to AB and equal to the figures AB and H together, and place its ſides AE, AF from the point A, upon the ſtraight lines AC, AD, and complete the parallelogram AG, which is about the ſame diameter [e] with AB. becauſe therefore AG is equal to both AB and H, take away the common part AB, the remaining gnomon ECBDFG is equal to the remaining figure H. therefore a gnomon equal to H, and its ſides CE, DF are found. and in like manner they may be found in the other caſe, in which the given figure H muſt be leſs than the figure FE from which it is to be taken.

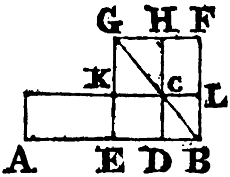
d. 25. 6.
e. 26. 6.

PROP. LXXXIII. 58.

IF a parallelogram equal to a given ſpace be applied to a given ſtraight line, deficient by a parallelogram given in ſpecies; the ſides of the defect are given.

Let the parallelogram AC equal to a given ſpace be applied to the given ſtraight line AB, deficient by the parallelogram BDCL given in ſpecies; each of the ſtraight lines CD, DB are given.

Biſect AB in E; therefore EB is given in magnitude. upon EB
deſcribe [a] the parallelogram EF ſimilar to DL and ſimilarly placed; a. 18. 6.
therefore EF is given in ſpecies, and is about
the ſame diameter [b] with DL; let BCG b. 26. 6.
be the diameter, and conſtruct the figure.
therefore becauſe the figure EF given in
ſpecies is deſcribed upon the given ſtraight
line EB, EF is given [c] in magnitude. and c. 56. Dat.
the gnomon ELH is equal [d] to the given d. 36. and 43. 1.
figure AC, therefore [e] ſince EF is diminiſhed by the given gno- e. 82. Dat.
mon ELH, the ſides EK, FH of the gnomon are given, but EK
is equal to DC, and FH to DB; wherefore CD, DB are each of
them given.



This Demonſtration is the Analyſis of the Problem in the 28. Prop. of Book 6. the conſtruction and Demonſtration of which Propoſition is the Compoſition of the Analyſis, and becauſe the given ſpace AC or its equal the gnomon ELH is to be taken from the figure EF deſcribed upon the half of AB ſimilar to BC, therefore AC muſt not be greater than EF, as is ſhewn in the 27. Prop. B. 6.

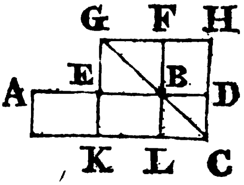
PROP. LXXXIV. 59.

IF a parallelogram equal to a given ſpace be applied to a given ſtraight line, exceeding by a parallelogram given in ſpecies; the ſides of the exceſs are given.

Let the parallelogram AC equal to a given ſpace be applied to the given ſtraight line AB, exceeding by the parallelogram BDCL given in ſpecies; each of the ſtraight lines CD, DB are given.

Biſect AB in E; therefore EB is given in magnitude. upon EB
deſcribe [a] the parallelogram EF ſimilar to LD, and ſimilarly placed; a. 18. 6.
therefore EF is given in ſpecies, and is about the ſame diameter [b] b. 26. 6.

with LD. let CBG be the diameter, and conſtruct the figure. therefore becauſe the figure EF given in ſpecies is deſcribed upon the given ſtraight line EB, EF is given
c. 56. Dat. in magnitude [c]. and the gnomon ELH is
d. 36. and equal to the given figure [d] AC; where-
43. 1. fore ſince EF is encreaſed by the given
e. 82. Dat. gnomon ELH, its ſides EK, FH are given [e]. but EK is equal to CD, and FH to BD; therefore CD, DB are each of them given.

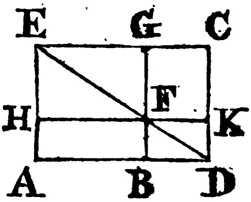


This Demonſtration is the Analyſis of the Problem in the 29. Prop. Book 6. the conſtruction and Demonſtration of which is the Compoſition of the Analyſis.

Cor. If a parallelogram given in ſpecies be applied to a given ſtraight line, exceeding by a parallelogram equal to a given ſpace; the ſides of the parallelogram are given.

Let the parallelogram ADCE given in ſpecies be applied to the given ſtraight line AB exceeding by the parallelogram BDCG equal to a given ſpace; the ſides AD, DC of the parallelogram are given.

Draw the diameter DE of the parallelogram AC, and conſtruct
a. 43. 1. the figure. becauſe the parallelogram AK is equal [a] to BC which is given, therefore AK is given. and BK is
b. 24. 6. ſimilar [b] to AC, therefore BK is given in ſpecies. and ſince the parallelogram AK given in magnitude is applied to the given ſtraight line AB, exceeding by the parallelogram BK given in ſpecies, therefore, by this Propoſition, BD, DK the ſides of the exceſs are given. and the ſtraight line AB is given, therefore the whole AD, as alſo DC to which it has a given ratio is given.



PROB.

To apply a parallelogram ſimilar to a given one to a given ſtraight line AB, exceeding by a parallelogram equal to a given ſpace.

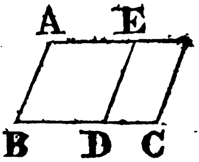
c. 29. 6. To the given ſtraight line AB apply [c] the parallelogram AK equal to the given ſpace, exceeding by the parallelogram BK ſimilar to the one given. draw DF the diameter of BK, and thro' the point A draw AE parallel to BF meeting DF produced in E, and complete the parallelogram AC.

The parallelogram BC is equal [a] to AK, that is to the given ſpace; and the parallelogram AC is ſimilar [b] to BK. therefore the parallelogram AC is applied to the ſtraight line AB ſimilar to the one given and exceeding by the parallelogram BC which is equal to the given ſpace.

PROP. LXXXV. 84.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the difference of the ſtraight lines be given, they ſhall each of them be given.

Let AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let the exceſs of BC above AB be given; each of the ſtraight lines AB, BC is given.



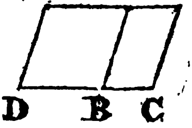
Let DC be the given exceſs of BC above BA, therefore the remainder BD is equal to BA. complete the parallelogram AD, and becauſe AB is equal to BD, the ratio of AB to BD is given. and the angle ABD is given, therefore the parallelogram AD is given in ſpecies. and becauſe the given parallelogram AC is applied to the given ſtraight line DC, exceeding by the parallelogram AD given in ſpecies, the ſides of the exceſs are given [a]; therefore BD is given. and DC is given, wherefore the whole BC is given. and AB is given, therefore AB, BC are each of them given. a. 84. Dat.

PROP. LXXXVI. 85.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if both of them together be given, they ſhall each of them be given.

Let the two ſtraight lines AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let AB, BC together be given; each of the ſtraight lines AB, BC is given.

Produce CB and make BD equal to BA, and complete the parallelogram ABDE. becauſe DB is equal to BA, and the angle ABD given, becauſe the adjacent angle ABC is given; the parallelogram AD is given in ſpecies. and becauſe AB, BC together are given, and AB is equal to BD; therefore DC is given. and becauſe the given parallelogram AC is applied to the given ſtraight line DC, deficient by the parallelogram AD given in ſpecies, the ſides AB, BD of the defect are given [a]. and DC is given, wherefore the remainder BC is given; and each of the ſtraight lines AB, BC is therefore given. a. 83. Dat.



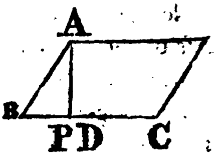
87.

PROP. LXXXVII.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the exceſs of the ſquare of the greater above the ſquare of the leſſer be given, each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC; if the exceſs of the ſquare of BC above the ſquare of BA be given; AB and BC are each of them given.

Let the given exceſs of the ſquare of BC above the ſquare of BA
be the rectangle CB, BD; take this from the ſquare of BC, the re-
a. 2. 2. mainder, which is [a] the rectangle BC, CD is equal to the ſquare of
AB. and becauſe the angle ABC of the parallelogram AC is given,
the ratio of the rectangle of the ſides AB, BC to the parallelogram
b. 62. Dat. AC is given [b]; and AC is given, therefore the rectangle AB, BC is
given; and the rectangle CB, BD is given; therefore the ratio of
c. 1. 6. the rectangle CB, BD to the rectangle AB, BC, that is [c] the ratio
d. 54. Dat. of the ſtraight line DB to BA is given; therefore [d] the ratio of the
ſquare of DB to the ſquare of BA is given.
and the ſquare of BA is equal to the rec-
tangle BC, CD; wherefore the ratio of the
rectangle BC, CD to the ſquare of BD is
given, as alſo the ratio of four times the rec-



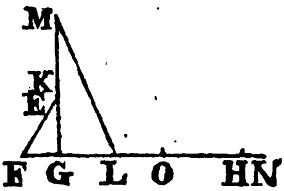
e. 7. Dat. tangle BC, CD to the ſquare of BD; and, by compoſition [e], the
ratio of four times the rectangle BC, CD together with the ſquare
of BD to the ſquare of BD is given. but four times the rectangle
f. 8. 2. BC, CD together with the ſquare of BD is equal [f] to the ſquare
of the ſtraight lines BC, CD taken together; therefore the ratio
of the ſquare of BC, CD together to the ſquare of BD is given;
g. 53. Dat. wherefore [g] the ratio of the ſtraight line BC together with CD to
BD is given. and, by compoſition, the ratio of BC together with
CD and DB, that is the ratio of twice BC to BD is given; there-
fore the ratio of BC to BD is given, as alſo [c] the ratio of the
ſquare of BC to the rectangle CB, BD. but the rectangle CB, BD
is given, being the given exceſs of the ſquares of BC, BA; there-
fore the ſquare of BC, and the ſtraight line BC is given. and the
ratio of BC to BD, as alſo of BD to BA has been ſhewn to be
h. 9. Dat. given; therefore [h] the ratio of BC to BA is given; and BC is
given, wherefore BA is given.

The preceding Demonſtration is the Analyſis of this Problem, viz.

A parallelogram AC which has a given angle ABC being given in magnitude, and the exceſs of the ſquare of BC one of its ſides above the ſquare of the other BA being given; to find the ſides. and

The Compoſition is as follows,

Let EFG be the given angle to which the angle ABC is required to be equal, and from any point E in FE draw EG perpendicular to FG; let the rectangle EG, GH be the given ſpace to which the parallelogram AC is to be made equal; and the rectangle HG, GL be the given exceſs of the ſquares of BC, BA.

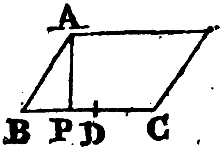


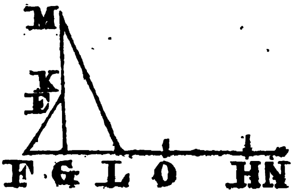
Take, in the ſtraight line GE, GK equal to FE, and make GM double of GK; join ML, and in GL produced take LN equal to LM. biſect GN in O, and between GH, GO find a mean proportional BC. as OG to GL, ſo make CB to BD; and make the angle CBA equal to GFE, and as LG to GK, ſo make DB to BA; and complete the parallelogram AC. AC is equal to the rectangle EG, GH, and the exceſs of the ſquares of CB, BA is equal to the rectangle HG, GL.

Becauſe as CB to BD, ſo is OG to GL, the ſquare of CB is to the rectangle CB, BD, as [a] the rectangle HG, GO to the rectangle HG, GL. and the ſquare of CB is equal to the rectangle HG, GO, becauſe GO, BC, GH are proportionals; therefore the rectangle CB, BD is equal [b] to HG, GL. and becauſe as CB to BD, ſo is OG to GL, twice CB is to BD, as twice OG, that is GN, to GL; and, by diviſion, as BC together with CD is to BD, ſo is NL, that is LM, to LG. therefore [c] the ſquare of BC together with CD is to the ſquare of BD, as the ſquare of ML to the ſquare of LG. but the ſquare of BC and CD together is equal [d] to four times the rectangle BC, CD together with the ſquare of BD; therefore four times the rectangle BC, CD together with the ſquare of BD is to the ſquare of BD, as the ſquare of ML to the ſquare of LG. and, by diviſion, four times the rectangle BC, CD is to the ſquare of BD, as the ſquare of MG to the ſquare of GL; wherefore the rectangle BC, CD is to the ſquare of BD, as (the ſquare of KG the half of MG to the ſquare of GL, that is as) the ſquare of AB to the ſquare of BD, becauſe as LG to GK, ſo DB was made to BA. therefore [b] the rectangle BC, CD is equal to the ſquare of AB; to each of theſe

a. 1. 6.
b. 14. 5.
c. 22. 6.
d. 8. 2.

add the rectangle CB, BD, and the ſquare of BC becomes equal to the ſquare of AB together with the rectangle CB, BD. therefore this rectangle, that is the given rectangle HG, GL is the exceſs of the ſquares of BC, AB from the point A draw AP perpendicular to BC, and becauſe the angle ABP is equal to the angle EFG, the triangle ABP is equiangular to EFG. and DB was made to BA, as LG to GK, therefore as the rectangle CB, BD to CB, BA, ſo is the rectangle HG, GL to HG, GK; and as the rectangle CB, BA to





AP, BC, ſo is (the ſtraight line BA to AP, and ſo is FE or GK to EG, and ſo is) the rectangle HG, GK to HG, GE; therefore, ex aequali, as the rectangle CB, BD to AP, BC, ſo is the rectangle HG, GL to EG, GH. and the rectangle CB, BD is equal to HG, GL, therefore the rectangle AP, BC, that is the parallelogram AC is equal to the given rectangle EG, GH.

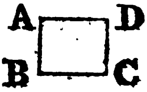
N.

PROP. LXXXVIII.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the ſum of the ſquares of its ſides be given, the ſides ſhall each of them be given.

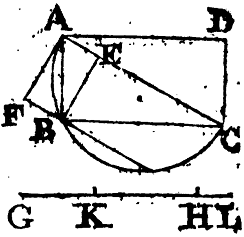
Let the two ſtraight lines AB, BC contain the parallelogram ABCD given in magnitude in the given angle ABC, and let the ſum of the ſquares of AB, BC be given; AB, BC are each of them given.

Firſt, let ABC be a right angle; and becauſe twice the rectangle contained by two equal ſtraight lines is equal to both their ſquares; but if two ſtraight lines are unequal, twice the rectangle contained by them is leſs than the ſum of their ſquares, as is evident from the 7. Prop. B. 2. Elem. therefore twice the given ſpace, to which ſpace the rectangle of which the ſides are to be found, is equal, muſt not be greater than the given ſum of the ſquares of the ſides. and if twice



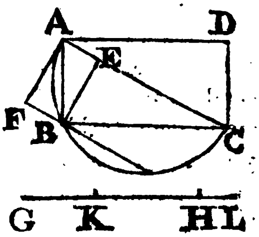
that ſpace be equal to the given ſum of the ſquares, the ſides of the rectangle muſt neceſſarily be equal to one another. therefore in this caſe deſcribe a ſquare ABCD equal to the given rectangle, and its ſides AB, BC are thoſe which were to be found. for the rectangle AC is equal to the given ſpace, and the ſum of the ſquares of its ſides AB, BC is equal to twice the rectangle AC, that is, by the hypotheſis, to the given ſpace to which the ſum of the ſquares was required to be equal.

But if twice the given rectangle be not equal to the given ſum
of the ſquares of the ſides, it muſt be leſs than it, as has been
ſhewn. Let ABCD be the rectangle, join AC and draw BE per-
pendicular to it, and complete the rectangle AEBF, and deſcribe
the circle ABC about the triangle ABC; AC is its diameter [a]. a. Cor. 5. 4.
and becauſe the triangle ABC is ſimilar [b] to AEB, as AC to CB, b. 8. 6.
ſo is AB to BE; therefore the rectangle AC, BE is equal to AB,
BC; and the rectangle AB, BC is given, wherefore AC, BE is
given. and becauſe the ſum of the ſquares of AB, BC is given,
the ſquare of AC which is equal [c] to that ſum is given; and AC c. 47. 1.
itſelf is therefore given in magnitude. let AC be likewiſe given
in poſition, and the point A; therefore AF
is given [d] in poſition. and the rectangle AC, d. 32. Dat.
BE is given, as has been ſhewn, and AC
is given, wherefore [e] BE is given in mag- e. 61. Dat.
nitude, as alſo AF which is equal to it;
and AF is alſo given in poſition, and the
point A is given; wherefore [f] the point F f. 30. Dat.
is given, and the ſtraight line FB in poſi-
tion [g]. and the circumference ABC is given in poſition, wherefore g. 31. Dat.
[h] the point B is given. and the points A, C are given; therefore h. 28. Dat.
the ſtraight lines AB, BC are given [i] in poſition and magni- i. 29. Dat.
tude.



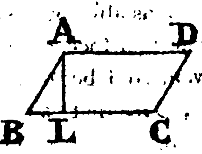
The ſides AB, BC of the rectangle may be found thus; let the
rectangle GH, GK be the given ſpace to which the rectangle AB,
BC is equal; and let GH, GL be the given rectangle to which the
ſum of the ſquares of AB, BC is equal. find [k] a ſquare equal to k. 14. 2.
the rectangle GH, GL, and let its ſide AC be given in poſition;
upon AC as a diameter deſcribe the ſemicircle ABC, and as AC to
GH, ſo make GK to AF, and from the point A place AF at right
angles to AC. therefore the rectangle CA, AF is equal [l] to GH, l. 16. 6.
GK; and, by the hypotheſis, twice the rectangle GH, GK is leſs

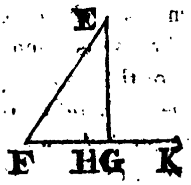
than GH, GL, that is than the ſquare of AC; wherefore twice the rectangle CA, AF is leſs than the ſquare of AC, and the rectangle CA, AF itſelf leſs than half the ſquare of AC, that is than the rectangle contained by the diameter AC and its half; wherefore AF is leſs than the ſemidiameter of the circle, and conſequently the ſtraight line drawn through the point F parallel to AC muſt meet the circumference in two points. let B be either of them, and join AB, BC and complete the rectangle ABCD; ABCD is the rectangle which was to be found. draw BE perpendicular to AC; therefore BE is a. 34. 1. equal [a] to AF, and becauſe the angle ABC in a ſemicircle is a b. 8. 6. right angle, the rectangle AB, BC is equal [b] to AC, BE, that is to the rectangle CA, AF which is equal to the given rectangle c. 47. 1. GH, GK, and the ſquares of AB, BC are together equal [c] to the ſquare of AC, that is to the given rectangle GH, GL.



But if the given angle ABC of the parallelogram AC be not a right angle, in this caſe becauſe ABC is a given angle, the ratio of the rectangle contained by the ſides AB, BC to the parallelogram n. 62. Dat. AC is given [n]; and AC is given, therefore the rectangle AB, BC is given. and the ſum of the ſquares of AB, BC is given; therefore the ſides AB, BC are given by the preceding caſe.

The ſides AB, BC and the parallelogram AC may be found thus. let EFG be the given angle of the parallelogram, and from any point E in FE draw EG perpendicular to FG. and let the rectangle EG, FH be the given ſpace to which the parallelogram is to be made equal, and let EF, FK be the given rectangle to which the ſum of the ſquares of the ſides is to be equal. and, by the preceding caſe, find the ſides of a rectangle which is equal to the given rectangle EF, FH, and the ſquares of the ſides of which are together equal to the given rectangle EF, FK. therefore, as was ſhewn in that caſe, twice the rectangle EF, FH muſt not be greater than the rectangle EF, FK; let it be ſo, and let AB, BC be the ſides of the rectangle joined in the angle ABC equal to the given angle EFG; and complete the parallelogram ABCD, which will be that which was to be found. draw AL perpendicular to





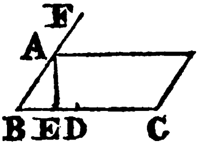
BC, and becauſe the angle ABL is equal to EFG, the triangle ABL is equiangular to EFG. and the parallelogram AC, that is the rectangle AL, BC is to the rectangle AB, BC as (the ſtraight line AL to AB, that is as EG to EF, that is as) the rectangle EG, FH to EF, FH. and, by the conſtruction, the rectangle AB, BC is equal to EF, FH, therefore the rectangle AL, BC, or, its equal, the parallelogram AC is equal to the given rectangle EG, FH. and the ſquares of AB, BC are together equal, by conſtruction, to the given rectangle EF, FK.

PROP. LXXXIX. 86.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the exceſs of the ſquare of one of them above a given ſpace has a given ratio to the ſquare of the other; each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the exceſs of the ſquare of BC above a given ſpace have a given ratio to the ſquare of AB; each of the ſtraight lines AB, BC is given.

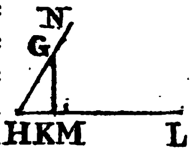
Becauſe the exceſs of the ſquare of BC above a given ſpace has a
given ratio to the ſquare of BA, let the rectangle CB, BD be the
given ſpace; take this from the ſquare of BC, the remainder, to wit,
the rectangle [a] BC, CD has a given ratio to the ſquare of BA. draw a. 2. 2.
AE perpendicular to BC, and let the ſquare of BF be equal to the
rectangle BC, CD. then becauſe the angle
ABC, as alſo BEA is given, the triangle ABE is
given [b] in ſpecies, and the ratio of AE to AB b. 43. Dat.
given. and becauſe the ratio of the rectangle
BC, CD, that is of the ſquare of BF to the
ſquare of BA is given, the ratio of the ſtraight line BF to BA is given [c]. and the ratio of AE to AB is given, wherefore [d] the ratio of c. 58. Dat.
AE to BF is given, as alſo the ratio of the rectangle AE, BC, that d. 9. Dat.
is [e] of the parallelogram AC to the rectangle FB, BC; and AC is e. 35. 1.
given, wherefore the rectangle FB, BC is given. and the exceſs of
the ſquare of BC above the ſquare of BF, that is above the rectangle BC, CD is given, for it is equal [a] to the given rectangle CB,
BD. therefore becauſe the rectangle contained by the ſtraight lines
FB, BC is given, and alſo the exceſs of the ſquare of BC above the



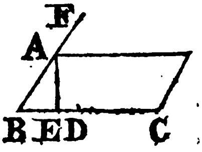
f. 87. Dat. ſquare of BF; FB, BC are each of them given [f]. and the ratio of FB to BA is given; therefore AB, BC are given.

The Compoſition is as follows.

Let GHK be the given angle to which the angle of the parallelogram is to be made equal, and from any point G in HG draw GK perpendicular to HK; let GK, HL be the rectangle to which the parallelogram is to be made equal, and let LH, HM be the rectangle equal to the given ſpace which is to be taken from the ſquare of one of the ſides; and let the ratio of the remainder to the ſquare of the other ſide be the ſame with the ratio of the ſquare of the given ſtraight line NH to the ſquare of the given ſtraight line HG.



By help of the 87. Dat. find two ſtraight lines BC, BF which contain a rectangle equal to the given rectangle NH, HL, and ſuch that the exceſs of the ſquare of BC above the ſquare of BF be equal to the given rectangle LH, HM; and join CB, BF in the angle FBC equal to the given angle GHK. and as NH to HG, ſo make FB to BA, and complete the parallelogram AC, and draw AE perpendicular to BC. then AC is equal to the rectangle GK, HL; and if from the ſquare of BC, the given rectangle LH, HM be taken, the remainder ſhall have to the ſquare of BA the ſame ratio which the ſquare of NH has to the ſquare of HG.



Becauſe, by the conſtruction, the ſquare of BC is equal to the ſquare of BF together with the rectangle LH, HM; if from the ſquare of BC there be taken the rectangle LH, HM, there remains
g. 22. 6. the ſquare of BF which has [g] to the ſquare of BA the ſame ratio which the ſquare of NH has to the ſquare of HG, becauſe as NH to HG, ſo FB was made to BA; but as HG to GK, ſo is BA to AE, becauſe the triangle GHK is equiangular to ABE; there-
h. 1. 6. fore, ex aequali, as NH to GK, ſo is FB to AE. wherefore [h] the rectangle NH, HL is to the rectangle GK, HL, as the rectangle FB, BC to AE, BC. but, by the conſtruction, the rectangle NH,
k. 14. 5. HL is equal to FB, BC; therefore [k] the rectangle GK, HL is equal to the rectangle AE, BC, that is to the parallelogram AC.

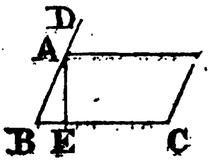
The Analyſis of this Problem might have been made as in the 86. Prop. in the Greek, and the compoſition of it may be made as that which is in Prop. 87. of this Edition.

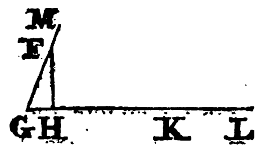
PROP. XC. O.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the ſquare of one of them be given together with the ſpace which has a given ratio to the ſquare of the other; each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the ſquare of BC be given together with the ſpace which has a given ratio to the ſquare of AB; AB, BC are each of them given.

Let the ſquare of BD be the ſpace which has the given ratio to the ſquare of AB; therefore, by the hypotheſis, the ſquare of BC together with the ſquare of BD is given. from the point A draw AE perpendicular to BC, and becauſe the angles ABE, BEA are given, the triangle ABE is given [a] in ſpecies; therefore the ratio of BA to AE is given. and becauſe the ratio of the ſquare of BD to the ſquare of BA is given, the ratio of the ſtraight line BD to BA is given [b]; and the ratio of BA to AE is given, therefore [c] the ratio of AE to BD is given, as alſo the ratio of the rectangle AE, BC, that is of the parallelogram AC to the rectangle DB, BC. and AC is given, therefore the rectangle DB, BC is given; and





the ſquare of BC together with the ſquare of BD is given. therefore [d] becauſe the rectangle contained by the two ſtraight lines DB, BC is given, and the ſum of their ſquares is given; the ſtraight lines DB, BC are each of them given. and the ratio of DB to BA is given; therefore AB, BC are given.

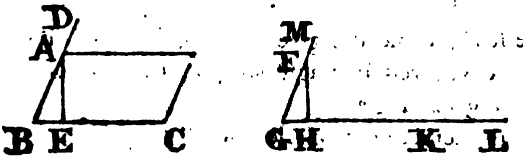
The Compoſition is as follows.

Let FGH be the given angle to which the angle of the parallelogram is to be made equal, and from any point F in GF draw FH perpendicular to GH; and let the rectangle FH, GK be that to which the parallelogram is to be made equal; and let the rectangle KG, GL be the ſpace to which the ſquare of one of the ſides of the

[a] 43. Dat.
[b] 58. Dat.
[c] 9. Dat.
[d] 88. Dat.

parallelogram together with the ſpace which has a given ratio to the ſquare of the other ſide, is to be made equal; and let this given ratio be the ſame which the ſquare of the given ſtraight line MG has to the ſquare of GF.

By the 88. Dat. find two ſtraight lines DB, BC which contain a rectangle equal to the given rectangle MG, GK, and ſuch that the ſum of their ſquares is equal to the given rectangle KG, GL. therefore, by the determination of the Problem in that Propoſition, twice the rectangle MG, GK muſt not be greater than the rectangle KG, GL. let it be ſo, and join the ſtraight lines DB, BC in the angle DBC equal to the given angle FGH. and as MG to GF, ſo make DB to BA, and complete the parallelogram AC, AC is equal to the rectangle FH, GK; and the ſquare of BC together with the ſquare of BD which by the conſtruction has to the ſquare of BA the given ratio which the ſquare of MG has to the ſquare of GF, is equal, by the conſtruction, to the given rectangle KG, GL. Draw AE perpendicular to BC.



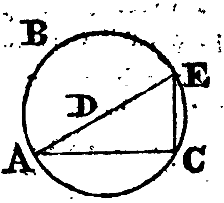
Becauſe as DB to BA, ſo is MG to GF; and as BA to AE, ſo GF to FH; ex aequali, as DB to AE, ſo is MG to FH. therefore as the rectangle DB, BC to AE, BC, ſo is the rectangle MG, GK to FH, GK. and the rectangle DB, BC is equal to the rectangle MG, GK; therefore the rectangle AE, BC, that is the parallelogram AC, is equal to the rectangle FH, GK.

88. PROP. XCI.

IF a ſtraight line drawn within a circle given in magnitude cuts off a ſegment which contains a given angle; the ſtraight line is given in magnitude.

In the circle ABC given in magnitude, let the ſtraight line AC be drawn cutting off the ſegment AEC which contains the given angle AEC; the ſtraight line AC is given in magnitude.

a. 1. 3. Take D the center of the circle [a], join AD and produce it to



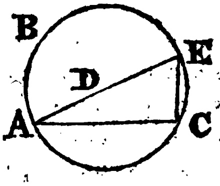
E, and join EC. the angle ACE being a right [b] angle is given; and the angle AEC is given; therefore [c] the triangle ACE is given in ſpecies, and the ratio of EA to AC is therefore given. and EA is given [d] in magnitude, becauſe the circle is given in magnitude; AC is therefore given [e] in magnitude.

b. 31. 3.
c. 43. Dat.
d. 5. Def.
e. 2. Dat.

PROP. XCII. 89.

IF a ſtraight line given in magnitude be drawn within a circle given in magnitude; it ſhall cut off a ſegment containing a given angle.

Let the ſtraight line AC given in magnitude be drawn within the circle ABC given in magnitude; it ſhall cut off a ſegment containing a given angle.



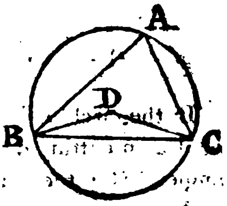
Take D the center of the circle, join AD and produce it to E, and join EC. and becauſe each of the ſtraight lines EA, AC is given, their ratio is given [a]; and the angle ACE is a right angle, therefore the triangle ACE is given [b] in ſpecies, and conſequently the angle AEC is given.

a. 1. Dat.
b. 46. Dat.

PROP. XCIII. 90.

IF from any point in the circumference of a circle given in poſition two ſtraight lines be drawn meeting the circumference and containing a given angle; if the point in which one of them meets the circumference again be given, the point in which the other meets it is alſo given.

From any point A in the circumference of a circle ABC given in poſition, let AB, AC be drawn to the circumference, making the given angle BAC; if the point B be given, the point C is alſo given.



Take D the center of the circle, and join BD, DC. and becauſe each of the points B, D is given, BD is given [a] in poſition; and becauſe the angle BAC is given, the angle BDC is given [b]. therefore becauſe the ſtraight line

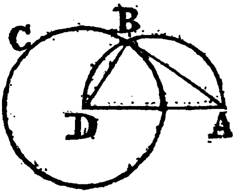
a. 29. Dat.
b. 20. 3.

DC is drawn to the given point D in the ſtraight line BD given
c. 32. Dat. in poſition in the given angle BDC, DC is given [c] in poſition. and
d. 28. Dat. the circumference ABC is given in poſition, therefore [d] the point
C is given.

91. PROP. XCIV.

IF from a given point a ſtraight line be drawn touching a circle given in poſition; the ſtraight line is given in poſition and magnitude.

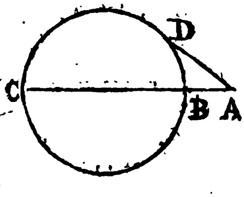
Let the ſtraight line AB be drawn from the given point A touching the circle BC given in poſition; AB is given in poſition and magnitude.



Take D the center of the circle, and join DA, DB. becauſe each
of the points D, A is given, the ſtraight
a. 29. Dat line AD is given [a] in poſition and magni-
b. 18. 3. tude. and DBA is a right [b] angle, where-
c. Cor. 5. 4. fore DA is a diameter [c] of the circle DBA
deſcribed about the triangle DBA; and
d. 6. Def. that circle is therefore given [d] in poſition.
and the circle BC is given in poſition, there-
e. 28. Dat. fore the point B is given [e]. the point A is alſo given; therefore
the ſtraight line AB is given [a] in poſition and magnitude.

92. PROP. XCV.

IF a ſtraight line be drawn from a given point without a circle given in poſition; the rectangle contained by the ſegments betwixt the point and the circumference of the circle is given.

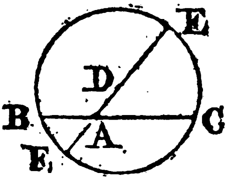


Let the ſtraight line ABC be drawn from the given point A
without the circle BCD given in poſition,
cutting it in B, C; the rectangle BA, AC
is given.
a. 17. 3. From the point A draw [a] AD touching
b. 94. Dat. the circle; therefore AD is given [b] in po-
ſition and magnitude. and becauſe AD is
c. 56. Dat. given, the ſquare of AD is given [c] which
d. 36. 3. is equal [d] to the rectangle BA, AC. therefore the rectangle BA,
AC is given.

PROP. XCVI. 93.

IF a ſtraight line be drawn thro' a given point within a circle given in poſition, the rectangle contained by the ſegments betwixt the point and the circumference of the circle is given.

Let the ſtraight line BAC be drawn thro' the given point A within the circle BCE given in poſition; the rectangle BA, AC is given.



Take D the center of the circle, join AD and produce it to the points E, F. becauſe the points A, D are given, the ſtraight line AD is given [a] in poſition; and the circle BEC is given in poſition; therefore the points E, F are given [b]. and the point A is given, therefore EA, AF are each of them given [a]; and the rectangle EA, AF is therefore given; and it is equal [c] to the rectangle BA, AC which conſequently is given.

a. 29. Dat.

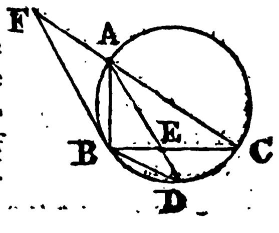
b. 28. Dat.

c. 35. 3.

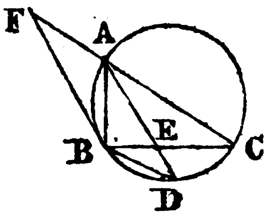
PROP. XCVII. 94.

IF a ſtraight line be drawn within a circle given in magnitude cutting off a ſegment containing a given angle; if the angle in the ſegment be biſected by a ſtraight line produced till it meets the circumference, the ſtraight lines which contain the given angle ſhall both of them together have a given ratio to the ſtraight line which biſects the angle. and the rectangle contained by both theſe lines together which contain the given angle, and the part of the biſecting line cut off below the baſe of the ſegment, ſhall be given.

Let the ſtraight line BC be drawn within the circle ABC given in magnitude cutting off a ſegment containing the given angle BAC. and let the angle BAC be biſected by the ſtraight line AD; BA together with AC has a given ratio to AD; and the rectangle contained by BA and AC together, and the ſtraight line ED cut off from AD below BC the baſe of the ſegment, is given.



Join BD; and becauſe BC is drawn within the circle ABC given in magnitude cutting off the ſegment BAC containing the given a. 91. Dat. angle BAC; BC is given [a] in magnitude. by the ſame reaſon BD b. 1. Dat. is given; therefore [b] the ratio of BC to BD is given. and becauſe c. 3. 6. the angle BAC is biſected by AD, as BA to AC, ſo is [c] BE to EC; d. 12. 5. and, by permutation, as AB to BE, ſo is AC to CE; wherefore [d] as BA and AC together to BC, ſo is AC to CE. and becauſe the angle BAE is equal to EAC, and the e. 21. 3. angle ACE to [e] ADB; the triangle ACE is equiangular to the triangle ADB; therefore as AC to CE, ſo is AD to DB. but as AC to CE, ſo is BA together with AC to BC; as therefore BA and AC to BC, ſo is AD to DB; and, by permutation, as BA and AC to AD, ſo is BC to BD. and the ratio of BC to BD is given, therefore the ratio of BA together with AC to AD is given.



Alſo the rectangle contained by BA and AC together, and DE is given.

Becauſe the triangle BDE is equiangular to the triangle ACE, as BD to DE, ſo is AC to CE; and as AC to CE, ſo is BA and AC to BC; therefore as BA and AC to BC, ſo is BD to DE. wherefore the rectangle contained by BA and AC together, and DE is equal to the rectangle CB, BD. but CB, BD is given; therefore the rectangle contained by BA and AC together, and DE is given.

Otherwiſe.

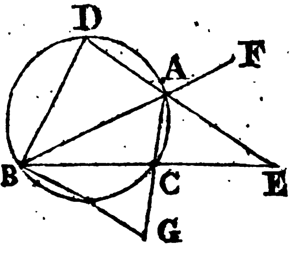
Produce CA and make AF equal to AB, and join BF. and be-a. 5. and 32. 1. cauſe the angle BAC is double [a] of each of the angles BFA, BAD, the angle BFC is equal to BAD; and the angle BCA is equal to BDA, therefore the triangle FCB is equiangular to ADB. as therefore FC to CB, ſo is AD to DB, and, by permutation, as FC, that is BA and AC together to AD, ſo is CB to BD. and the ratio of CB to BD is given, therefore the ratio of BA and AC to AD is given.

And becauſe the angle BFC is equal to the angle DAC, that is to the angle DBC, and the angle ACB equal to the angle ADB; the triangle FCB is equiangular to BDE, as therefore FC to CB, ſo is BD to DE; therefore the rectangle contained by FC, that is BA and AC together, and DE is equal to the rectangle CB, BD which is given, and therefore the rectangle contained by BA, AC together, and DE is given.

PROP. XCVIII.

IF a ſtraight line be drawn within a circle given in magnitude cutting off a ſegment containing a given angle; if the angle adjacent to the angle in the ſegment be biſected by a ſtraight line produced till it meet the circumference again and the baſe of the ſegment; the exceſs of the ſtraight lines which contain the given angle ſhall have a given ratio to the ſegment of the biſecting line which is within the circle; and the rectangle contained by the ſame exceſs and the ſegment of the biſecting line betwixt the baſe produced and the point where it again meets the circumference, ſhall be given.

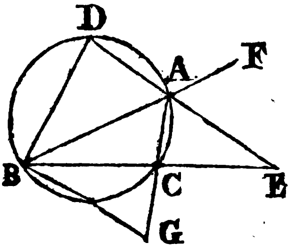
Let the ſtraight line BC be drawn within the circle ABC given in magnitude cutting off a ſegment containing the given angle BAC, and let the angle CAF adjacent to BAC be biſected by the ſtraight line DAE meeting the circumference again in D, and BC the baſe of the ſegment produced in E; the exceſs of BA, AC has a given ratio to AD; and the rectangle which is contained by the ſame exceſs and the ſtraight line ED, is given.



Join BD, and thro' B draw BG parallel to DE meeting AC produced in G. and becauſe BC cuts off from the circle ABC given in magnitude the ſegment BAC containing a given angle, BC is therefore given [a] in magnitude. by the ſame reaſon BD is given, becauſe the angle BAD is equal to the given angle EAF; therefore the ratio of BC to BD is given. and becauſe the angle CAE is equal to EAF, of which CAE is equal to the alternate angle AGB, and EAF to the interior and oppoſite angle ABG; therefore the angle AGB is equal to ABG, and the ſtraight line AB equal to AG; ſo that GC is the exceſs of BA, AC. and becauſe the angle BGC is equal to GAE, that is to EAF, or the angle BAD; and that the angle BCG is equal to the oppoſite interior angle BDA of the quadrilateral BCAD in the circle; therefore the triangle BGC is equiangular to BDA.

a. 91. Dat.

therefore as GC to CB, ſo is AD to DB, and, by permutation, as GC, which is the exceſs of BA, AC to AD, ſo is CB to BD. and the ratio of CB to BD is given; therefore the ratio of the exceſs of BA, AC to AD is given.



And becauſe the angle GBC is equal to the alternate angle DEB, and the angle BCG equal to BDE; the triangle BCG is equiangular to BDE. therefore as GC to CB, ſo is BD to DE, and conſequently the rectangle GC, DE is equal to the rectangle CB, BD which is given, becauſe its ſides CB, BD are given. therefore the rectangle contained by the exceſs of BA, AC and the ſtraight line DE is given.

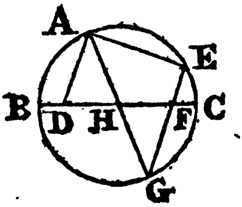
95.
PROP. XCIX.

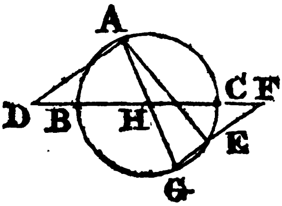
IF from a given point in the diameter of a circle given in poſition, or in the diameter produced, a ſtraight line be drawn to any point in the circumference, and from that point a ſtraight line be drawn at right angles to the firſt, and from the point in which this meets the circumference again, a ſtraight line be drawn parallel to the firſt; the point in which this parallel meets the diameter is given; and the rectangle contained by the two parallels is given.

In BC the diameter of the circle ABC given in poſition, or in BC produced, let the given point D be taken, and from D let a ſtraight line DA be drawn to any point A in the circumference, and let AE be drawn at right angles to DA, and from the point E where it meets the circumference again let EF be drawn parallel to DA meeting BC in F; the point F is given, as alſo the rectangle AD, EF.

Produce EF to the circumference in G, and join AG. becauſe
a. Cor. 5. 4. GEA is a right angle, the ſtraight line AG is [a] the diameter of the circle ABC; and BC is alſo a diameter of it; therefore the point H where they meet is the center of the circle, and conſequently H is given. and the point D is given, wherefore DH is given

in magnitude. and becauſe AD is parallel to FG, and GH equal to HA; DH is equal [b] to HF, and AD equal to GF. and DH is b. 4. 6.





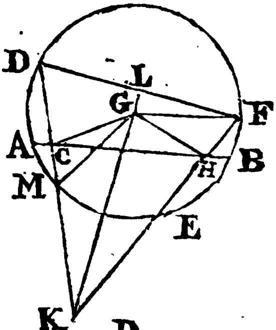
given, therefore HF is given in magnitude; and it is alſo given in poſition, and the point H is given, therefore [c] the point F is given. c. 30. Dat.

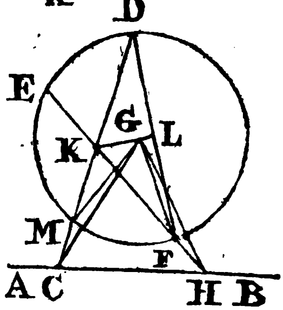
And becauſe the ſtraight line EFG is drawn from a given point F without or within the circle ABC given in poſition, therefore [d] the rectangle EF, FG is given. and GF is equal to AD, wherefore the rectangle AD, EF is given. d. 95. or 96. Dat.

PROP. C. Q.

IF from a given point in a ſtraight line given in poſition, a ſtraight line be drawn to any point in the circumference of a circle given in poſition; and from this point a ſtraight line be drawn making with the firſt an angle equal to the difference of a right angle and the angle contained by the ſtraight line given in poſition, and the ſtraight line which joins the given point and the center of the circle; and from the point in which the ſecond line meets the circumference again, a third ſtraight line be drawn making with the ſecond an angle equal to that which the firſt makes with the ſecond. the point in which this third line meets the ſtraight line given in poſition is given; as alſo the rectangle contained by the firſt ſtraight line and the ſegment of the third betwixt the circumference and the ſtraight line given in poſition, is given.

Let the ſtraight line CD be drawn from the given point C in the ſtraight line AB given in poſition, to the circumference of the circle DEF given in poſition of which G is the center; join CG, and from the point D let DF be drawn making the angle CDF equal to the difference of a right angle and the angle BCG, and from the point F let FE be drawn making the angle DFE equal to CDF, meeting AB in H. the point H is given; as alſo the rectangle CD, FH.





Let CD, FH meet one another in the point K from which draw KL perpendicular to DF; and let DC meet the circumference again in M, and let FH meet the ſame in E, and join MG, GF, GH.

Becauſe the angles MDF, DFE are equal to one another, the circumferences a. 26. 3. MF, DE are equal [a]; and adding or taking away the common part ME, the circumference DM is equal to EF; therefore the ſtraight line DM is equal to the ſtraight line EF, and the angle GMD to b. 8. 1. the angle [b] GFE; and the angles GMC, GFH are equal to one another, becauſe they are either the ſame with the angles GMD, GFE, or adjacent to them. and becauſe the angles KDL, LKD are together c. 32. 1. equal [c] to a right angle, that is, by the hypotheſis, to the angles KDL, GCB; the angle GCB or GCH is equal to the angle (LKD, that is to the angle) LKF or GKH. therefore the points C, K, H, G are in the circumference of a circle; and the angle GCK is therefore equal to the angle GHF; and the angle GMC is equal to GFH, and the ſtraight line GM to GF; d. 26. 1. therefore [d] CG is equal to GH, and CM to HF. and becauſe CG is equal to GH, the angle GCH is equal to GHC; but the angle GCH is given, therefore GHC is given; and conſequently the angle CGH is given. and CG is given in poſition, and the point e. 32. Dat. G; therefore [e] GH is given in poſition; and CB is alſo given in poſition, wherefore the point H is given.

And becauſe HF is equal to CM, the rectangle DC, FH is f. 95. or 96. Dat. equal to DC, CM. but DC, CM is given [f], becauſe the point C is given; therefore the rectangle DC, FH is given.

FINIS.

NOTES

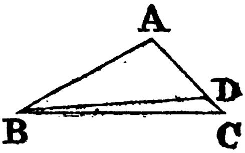
ON

EUCLID'S DATA.

DEFINITION II.

THIS is made more explicit than in the Greek text, to prevent a miſtake which the Author of the ſecond Demonſtration of the 24th Propoſition in the Greek Edition has fallen into, of thinking that a ratio is given to which another ratio is ſhewn to be equal, tho' this other be not exhibited in given magnitudes. See the Notes on that Propoſition which is the 13th in this Edition. beſides by this Definition, as it is now given, ſome Propoſitions are demonſtrated, which in the Greek are not ſo well done by help of Prop. 2.

DEF. IV.



In the Greek text Def. 4. is thus "Points, lines, ſpaces and "angles are ſaid to be given in poſition which have always the ſame "ſituation." but this is imperfect and uſeleſs, becauſe there are innumerable caſes in which things may be given according to this Definition, and yet their poſition cannot be found. for inſtance, let the triangle ABC be given in poſition, and let it be propoſed to draw a ſtraight line BD from the angle at B to the oppoſite ſide AC which ſhall cut off the angle DBC which ſhall be the ſeventh part of the angle ABC. ſuppoſe this is done, therefore the ſtraight line BD is invariable in its poſition, that is, has always the ſame ſituation; for any other ſtraight line drawn from the point B on either ſide of BD cuts off an angle greater or leſſer than the ſeventh part of the angle ABC; therefore, according to this Definition, the ſtraight line BD is given in poſition, as alſo [a] the point D in which it meets the ſtraight line AC which is given in poſition. but from the things here given, neither the ſtraight line BD nor the point D can be

[a] 28. Dat.

found by the help of Euclid's Elements only, by which every thing in his Data is ſuppoſed may be found. this Definition is therefore of no uſe. we have amended it by adding "and which are either "actually exhibited or can be found;" for nothing is to be reckoned given, which cannot be found, or is not actually exhibited.

The Definition of an angle given by poſition is taken out of the 4th, and given more diſtinctly by itſelf in the Definition marked A.

DEF. XI. XII. XIII. XIV. XV.

The 11th and 12th are omitted becauſe they cannot be given in Engliſh ſo as to have any tolerable ſenſe, and therefore wherever the terms defined occur, the words which expreſs their meaning are made uſe of in their place.

The 13. 14. 15. are omitted as being of no uſe.

It is to be obſerved in general of the Data in this book, that they are to be underſtood to be given Geometrically, not always Arithmetically, that is, they cannot always be exhibited in numbers; for inſtance, if the ſide of a ſquare be given, the ratio of it to its diameter is given [b] geometrically, but not in numbers; and the diameter is given [c], but tho' the number of any equal parts in the ſide be given, for example 10, the number of them in the diameter cannot be given. and the like holds in many other caſes.

b. 44. Dat.
c. 2. Dat.

PROPOSITION I.

In this it is ſhewn that A is to B, as C to D, from this that A is to C, as B to D, and then by permutation; but it follows directly, without theſe two ſteps, from 7. 5.

PROP. II.

The limitation added at the end of this Propoſition between the inverted commas is quite neceſſary, becauſe without it the Propoſition cannot always be demonſtrated. for the Author having ſaid * "becauſe A is given, a magnitude equal to it can be found [a], let "this be C; and becauſe the ratio of A to B is given, a ratio "which is the ſame to it can be found [b]" adds, "let it be found, "and let it be the ratio of C to Δ." Now from the ſecond Definition nothing more follows than that ſome ratio, ſuppoſe the ratio of E to Z, can be found, which is the ſame with the ratio of A to B; and when the Author ſuppoſes that the ratio of C to Δ, which is

a. 1. Def.
b. 2. Def.

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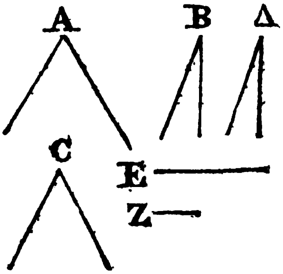
* See Dr. Gregory's Edition of the Data.

alſo the ſame with the ratio of A to B, can be found, he neceſſarily ſuppoſes that to the three magnitudes E, Z, C a fourth proportional Δ may be found; but this cannot always be done by the Elements of Euclid; from which it is plain Euclid muſt have underſtood the Propoſition under the limitation which is now added to his text. An example will make this clear; let A be a given angle, and B another angle to which A has a given ratio, for inſtance, the ratio of the given ſtraight line E to the given one Z, then, having found an angle C equal to A, how can the angle Δ be found to which C has the ſame ratio that E has to Z? certainly no way, until it be ſhewn how to find an angle to which a given angle has a given ratio, which cannot be done by Euclid's Elements, nor probably by any Geometry known in his time. Therefore in all the Propoſitions of this book which depend upon this ſecond, the above-mentioned limitation muſt be underſtood, tho' it be not explicitly mentioned.



PROP. V.

The order of the Propoſitions in the Greek text between Prop. 4. and Prop. 25. is now changed into another which is more natural, by placing thoſe which are more ſimple before thoſe which are more complex; and by placing together thoſe which are of the ſame kind, ſome of which were mixed among others of a different kind. thus Prop. 12. in the Greek is now made the 5. and thoſe which were the 22. and 23. are made the 11. and 12. as they are more ſimple than the Propoſitions concerning magnitudes the exceſs of one of which above a given magnitude has a given ratio to the other, after which theſe two were placed; and the 24. in the Greek text is, for the ſame reaſon, made the 13.

PROP. VI. VII.

Theſe are univerſally true, tho' in the Greek text they are demonſtrated by Prop. 2. which has a limitation; they are therefore now ſhewn without it.

PROP. XII.

In the 23. Prop. in the Greek text, which here is the 12. the words "μὴ τὰς αὐτὰς δέ" are wrong tranſlated by Claud. Hardy in his Edition of Euclid's Data printed at Paris Ann. 1625, which was the firſt Edition of the Greek text; and Dr. Gregory follows him in tranſlating them by the words "etſi non eaſdem," as if the Greek had been εἰ καὶ μὴ τὰς αὐτὰς as in Prop 9. of the Greek text. Euclid's meaning is that the ratios mentioned in the Propoſition muſt not be the ſame; for if they were, the Propoſition would not be true. whatever ratio the whole has to the whole, if the ratios of the parts of the firſt to the parts of the other be the ſame with this ratio, one part of the firſt may be double, triple, &c. of the other part of it, or have any other ratio to it, and conſequently cannot have a given ratio to it. wherefore theſe words muſt be rendered by "non autem eaſdem," but not the ſame ratios, as Zambertus has tranſlated them in his Edition.

PROP. XIII.

Some very ignorant Editor has given a ſecond Demonſtration of this Propoſition in the Greek text, which has been as ignorantly kept in it by Claud. Hardy and Dr. Gregory, and has been retained in the tranſlations of Zambertus and others; Carolus Renaldinus gives it only. the author of it has thought that a ratio was given if another ratio could be ſhewn to be the ſame to it, tho' this laſt ratio be not found. but this is altogether abſurd, becauſe from it would be deduced that the ratio of the ſides of any two ſquares is given, and the ratio of the diameters of any two circles, &c. and it is to be obſerved that the moderns frequently take given ratios, and ratios that are always the ſame for one and the ſame thing, and Sir Iſaac Newton has fallen into this miſtake in the 17th Lemma of his Principia, Ed. 1713, and in other places. but this ſhould be carefully avoided, as it may lead into other errors.

PROP. XIV. XV.

Euclid in this book has ſeveral Propoſitions concerning magnitudes, the exceſs of one of which above a given magnitude has a given ratio to the other; but he has given none concerning magnitudes whereof one together with a given magnitude has a given ratio to the other; tho' theſe laſt occur as frequently in the ſolution of Problems as the firſt. the reaſon of which is, that the laſt may be all demonſtrated by help of the firſt; for if a magnitude together with a given magnitude has a given ratio to another magnitude; the exceſs of this other above a given magnitude ſhall have a given ratio to the firſt, and on the contrary; as we have demonſtrated in Prop. 14. and for a like reaſon Prop. 15. has been added to the Data. one example will make the thing clear; ſuppoſe it were to be demonſtrated, That if a magnitude A together with a given magnitude has a given ratio to another magnitude B, that the two magnitudes A and B, together with a given magnitude have a given ratio to that other magnitude B; which is the ſame Propoſition with reſpect to the laſt kind of magnitudes above-mentioned, that the firſt part of Prop. 16. in this Edition is in reſpect of the firſt kind. this is ſhewn thus; from the hypotheſis, and by the firſt part of Prop. 14. the exceſs of B above a given magnitude has unto A a given ratio; and therefore, by the firſt part of Prop. 17. the exceſs of B above a given magnitude has unto B and A together a given ratio; and by the ſecond part of Prop. 14. A and B together with a given magnitude has unto B a given ratio; which is the thing that was to be demonſtrated. in like manner the other Propoſitions concerning the laſt kind of magnitudes may be ſhewn.

PROP. XVI. XVII.

In the third part of Prop. 10. in the Greek text, which is the 16. in this Edition, after the ratio of EC to CB has been ſhewn to be given; from this, by inverſion and converſion, the ratio of BC to BE is demonſtrated to be given; but, without theſe two ſteps, the concluſion ſhould have been made only by citing the 6. Propoſition. and in like manner, in the firſt part of Prop. 11. in the Greek, which in this Edition is the 17. from the ratio of DB to BC being given, the ratio of DC to DB is ſhewn to be given, by inverſion and Compoſition, inſtead of citing Prop. 7. and the ſame fault occurs in the ſecond part of the ſame Prop. 11.

PROP. XXI. XXII.

Theſe are now added, as being wanting to complete the ſubject treated of in the four preceding Propoſitions.

PROP. XXIII.

This which is Prop. 20. in the Greek text, was ſeparated from Prop. 14. 15. 16. in that text, after which it ſhould have been immediately placed, as being of the ſame kind. it is now put into its proper place. but Prop. 21. in the Greek is left out, as being the ſame with Prop. 14. in that text, which is here Prop. 18.

PROP. XXIV.

This, which is Prop. 13. in the Greek, is now put into its proper place, having been disjoined from the three following it in this Edition, which are of the ſame kind.

PROP. XXVIII.

This which in the Greek text is Prop. 25. and ſeveral of the following Propoſitions, are there deduced from Def. 4. which is not ſufficient, as has been mentioned in the Note on that Definition; they are therefore now ſhewn more explicitly.

PROP. XXXIV. XXXVI.

Each of theſe has a Determination, which is now added, which occaſions a change in their Demonſtrations.

PROP. XXXVII. XXXIX. XL. XLI.

The 35. and 36. Propoſitions in the Greek text are joined into one, which makes the 39. in this Edition, becauſe the ſame Enuntiation and Demonſtration ſerves both. and for the ſame reaſon Prop. 37. 38. in the Greek are joined into one which here is the 40.

Prop. 37. is added to the Data, as it frequently occurs in the ſolution of Problems. and Prop. 41. is added to complete the reſt.

PROP. XLII.

This is Prop. 39. in the Greek text, where the whole conſtruction of Prop. 22. of Book 1. of the Elements is put without need into the Demonſtration, but is now only cited.

PROP. XLV.

This is Prop. 42. in the Greek, where the three ſtraight lines made uſe of in the conſtruction are ſaid, but not ſhewn, to be ſuch that any two of them is greater than the third, which is now done.

PROP. XLVII.

This is Prop. 44. in the Greek text, but the Demonſtration of it is changed into another wherein the ſeveral caſes of it are ſhewn, which, tho' neceſſary, is not done in the Greek.

PROP. XLVIII.

There are two caſes in this Propoſition, ariſing from the two caſes of the 3d part of Prop. 47. on which the 48. depends. and in the Compoſition theſe two caſes are explicitly given.

PROP. LII.

The Conſtruction and Demonſtration of this which is Prop. 48. in the Greek, are made ſomething ſhorter than in that text.

PROP. LIII.

Prop. 63. in the Greek text is omitted, being only a caſe of Prop. 49. in that text, which is Prop. 53. in this Edition.

PROP. LVIII.

This is not in the Greek text, but its Demonſtration is contained in that of the firſt part of Prop. 54. in that text; which Propoſition is concerning figures that are given in ſpecies; this 58. is true of ſimilar figures, tho' they be not given in ſpecies, and as it frequently occurs, it was neceſſary to add it.

PROP. LIX. LXI.

This is the 54. in the Greek; and the 77. in the Greek, being the very ſame with it, is left out. and a ſhorter Demonſtration is given of Prop. 61.

PROP. LXII.

This which is moſt frequently uſeful is not in the Greek, and is neceſſary to Prop. 87. 88. in this Edition, as alſo, tho' not mentioned, to Prop. 86. 87. in the former Editions. Prop. 66. in the Greek text is made a Corollary to it.

PROP. LXIV.

This contains both Prop. 74. and 73. in the Greek text; the firſt caſe of the 74. is a repetition of Prop. 56. from which it is ſeparated in that text by many Propoſitions; and as there is no order in theſe Propoſitions, as they ſtand in the Greek, they are now put into the order which ſeemed moſt convenient and natural.

The Demonſtration of the firſt part of Prop. 73. in the Greek is groſsly vitiated. Dr. Gregory ſays that the ſentences he has incloſed betwixt two ſtars are ſuperfluous and ought to be cancelled; but he has not obſerved that what follows them is abſurd, being to prove that the ratio [ſee his figure] of ΑΓ to ΓΚ is given, which by the Hypotheſis at the beginning of the Propoſition is expreſsly given; ſo that the whole of this part was to be altered, which is done in this Prop. 64.

PROP. LXVII. LXVIII.

Prop. 70. in the Greek text is divided into theſe two, for the ſake of diſtinctneſs; and the Demonſtration of the 67. is rendered ſhorter than that of the firſt part of Prop. 70. in the Greek by means of Prop. 23. of Book 6. of the Elements.

PROP. LXX.

This is Prop. 62. in the Greek text; Prop. 78. in that text is only a particular caſe of it, and is therefore omitted.

Dr. Gregory in the Demonſtration of Prop. 62. cites the 49. Prop. Dat. to prove that the ratio of the figure AEB to the parallelogram AH is given, whereas this was ſhewn a few lines before; and beſides the 49. Prop. is not applicable to theſe two figures, becauſe AH is not given in ſpecies, but is, by the ſtep for which the citation is brought, proved to be given in ſpecies.

PROP. LXXIII.

Prop. 83. in the Greek text is neither well enuntiated nor demonſtrated. the 73. which in this Edition is put in place of it, is really the ſame, as will appear by conſidering [ſee Dr. Gregory's Edition] that A, B, Γ, E in the Greek text are four proportionals, and that the Propoſition is to ſhew that Δ, which has a given ratio to E, is to Γ, as B is to the ſtraight line to which A has a given ratio; or, by inverſion, that Γ is to Δ, as the ſtraight line to which A has a given ratio is to B; that is, if the proportionals be placed in this order, viz. Γ, E, A, B, that the firſt Γ is to Δ to which the ſecond E has a given ratio, as the ſtraight line to which the third A has a given ratio is to the fourth B; which is the Enuntiation of this 73. and was thus changed that it might be made like to that of Prop. 72. in this Edition, which is the 82. in the Greek text. and the Demonſtration of Prop. 73. is the ſame with that of Prop. 72. only making uſe of Prop. 23. inſtead of Prop. 22. of Book 5. of the Elements.

PROP. LXXVII.

This is put in place of Prop. 79. in the Greek text which is not a Datum, but a Theorem premiſed as a Lemma to Prop. 80. in that text. and Prop. 79. is made Cor. 1. to Prop. 77. in this Edition. Cl. Hardy in his Edition of the Data takes notice, that, in Prop. 80. of the Greek text, the parallel KL in the figure of Prop. 77. in this Edition muſt meet the circumference, but does not demonſtrate it, which is done here at the end of Cor. 3. of Prop. 77. in the conſtruction for finding a triangle ſimilar to ABC.

PROP. LXXVIII.

The Demonſtration of this which is Prop. 80. in the Greek is rendered a good deal ſhorter by help of Prop. 77.

PROP. LXXIX. LXXX. LXXXI.

Theſe are added to Euclid's Data, as Propoſitions which are often uſeful in the ſolution of Problems.

PROP. LXXXII.

This which is Prop. 60. in the Greek text is placed before the 83. and 84. which in the Greek are the 58. and 59. becauſe the Demonſtration of theſe two in this Edition is deduced from that of Prop. 82. from which they naturally follow.

PROP. LXXXVIII. XC.

Dr. Gregory in his preface to Euclid's Works which he publiſhed at Oxford in 1703, after having told that he had ſupplied the defects of the Greek text of the Data in innumerable places from ſeveral Manuſcripts, and corrected Cl. Hardy's tranſlation by Mr. Bernard's, adds, that the 86. Theorem "or Propoſition," ſeemed to be remarkably vitiated, but which could not be reſtored by help of the Manuſcripts; then he gives three different tranſlations of it in Latin, according to which he thinks it may be read; the two firſt have no diſtinct meaning, and the third which he ſays is the beſt, tho' it contains a true Propoſition which is the 90. in this Edition, has no connexion in the leaſt with the Greek text. and it is ſtrange that Dr. Gregory did not obſerve, that if Prop. 86. was changed into this, the Demonſtration of the 86. muſt be cancelled, and another put in its place. But, the truth is, both the Enuntiation and the Demonſtration of Prop. 86. are quite entire and right, only Prop. 87. which is more ſimple, ought to have been placed before it; and the deficiency which the Doctor juſtly obſerves to be in this part of Euclid's Data, and which no doubt is owing to the careleſſneſs and ignorance of the Greek Editors ſhould have been ſupplied, not by changing Prop. 86. which is both entire and neceſſary, but by adding the two Propoſitions which are the 88. and 90. in this Edition.

PROP. XCVIII. C.

Theſe were communicated to me by two excellent Geometers, the firſt of them by the Right Honourable the Earl Stanhope, and the other by Dr. Matthew Stewart; to which I have added the Demonſtrations.

Tho' the order of the Propoſitions has been in many places changed from that in former Editions, yet this will be of little diſadvantage, as the antient Geometers never cite the Data, and the Moderns very rarely.

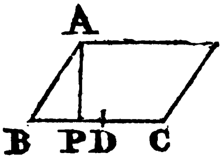
AS that part of the Compoſition of a Problem which is its Conſtruction may not be ſo readily deduced from the Analyſis by beginners; for their ſake the following Example is given in which the derivation of the ſeveral parts of the Conſtruction from the Analyſis is particularly ſhewn, that they may be aſſiſted to do the like in other Problems.

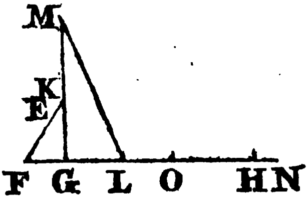
PROBLEM.

Having given the magnitude of a parallelogram, the angle of which ABC is given, and alſo the exceſs of the ſquare of its ſide BC above the ſquare of the ſide AB; To find its ſides and deſcribe it.

The Analyſis of this is the ſame with the Demonſtration of the 87. Prop. of the Data. and the Conſtruction that is given of the Problem at the end of that Propoſition, is thus derived from the Analyſis.

Let EFG be equal to the given angle ABC, and becauſe in the Analyſis it is ſaid that the ratio of the rectangle AB, BC to the parallelogram AC is given by the 62. Prop. Dat. therefore from a point in FE, the perpendicular EG is drawn to FG, as the ratio of FE to EG is the ratio of the rectangle AB, BC to the parallelogram AC by what is ſhewn at the end of Prop. 62. Next the magnitude of AC is exhibited by making the rectangle EG, GH equal to it, and the given exceſs of the ſquare of BC above the ſquare of BA, to which exceſs the rectangle CB, BD is equal, is exhibited by the rectangle HG, GL. then in the Analyſis the rectangle AB, BC is ſaid to be given, and this is equal to the rectangle FE, GH, becauſe the rectangle AB, BC is to the parallelogram AC, as (FE to EG, that is as the rectangle) FE, GH to EG, GH; and the parallelogram AC is equal to the rectangle EG, GH, therefore the rectangle AB, BC is equal to FE, GH. and conſequently the ratio of the rectangle CB, BD, that is of the rectangle HG, GL, to AB, BC, that





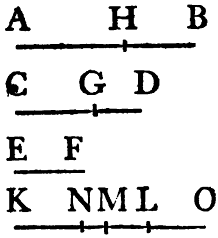
is of the ſtraight line DB to BA, is the ſame with the ratio (of the rectangle GL, GH to FE, GH, that is) of the ſtraight line GL to FE, which ratio of DB to BA is the next thing ſaid to be given in the Analyſis. from this it is plain that the ſquare of FE is to the ſquare of GL, as the ſquare of BA which is equal to the rectangle BC, CD is to the ſquare of BD, the ratio of which ſpaces is the next thing ſaid to be given. and from this it follows that four times the ſquare of FE is to the ſquare of GL, as four times the rectangle BC, CD is to the ſquare of BD; and, by Compoſition, four times the ſquare of FE together with the ſquare of GL is to the ſquare of GL, as four times the rectangle BC, CD together with the ſquare of BD, is to the ſquare of BD, that is [8. 6.] as the ſquare of the ſtraight lines BC, CD taken together is to the ſquare of BD, which ratio is the next thing ſaid to be given in the Analyſis. and becauſe four times the ſquare of FE and the ſquare of GL are to be added together, therefore in the perpendicular EG there is taken KG equal to FE, and MG equal to the double of it, becauſe thereby the ſquares of MG, GL, that is, joining ML, the ſquare of ML is equal to four times the ſquare of FE and to the ſquare of GL. and becauſe the ſquare of ML is to the ſquare of GL, as the ſquare of the ſtraight line made up of BC and CD is to the ſquare of BD, therefore [22. 6.] ML is to LG, as BC together with CD is to BD, and, by Compoſition, ML and LG together, that is, producing GL to N, ſo that ML be equal to LN, the ſtraight line NG is to GL, as twice BC is to BD; and by taking GO equal to the half of NG, GO is to GL, as BC to BD the ratio of which is ſaid to be given in the Analyſis. and from this it follows, that the rectangle HG, GO is to HG, GL, as the ſquare of BC is to the rectangle CB, BD which is equal to the rectangle HG, GL, and therefore the ſquare of BC is equal to the rectangle HG, GO, and BC is conſequently found by taking a mean proportional betwixt HG and GO, as is ſaid in the Conſtruction. and becauſe it was ſhewn that GO is to GL, as BC to BD, and that now the three firſt are found, the fourth BD is found by 12. 6. it was likewiſe ſhewn that LG is to FE, or GK, as DB to BA, and the three firſt are now found, and thereby the fourth BA. make the angle ABC equal to EFG, and complete the parallelogram of which the ſides are AB, BC, and the conſtruction is finiſhed; the reſt of the Compoſition contains the Demonſtration.

As the Propoſitions from the 13. to the 28. may be thought by beginners to be leſs uſeful than the reſt, becauſe they cannot ſo readily ſee how they are to be made uſe of in the ſolution of Problems; on this account the two following Problems are added, to ſhew that they are equally uſeful with the other Propoſitions, and from which it may eaſily be judged that many other Problems depend upon theſe Propoſitions.

PROBLEM I.

To find three ſtraight lines ſuch, that the ratio of the firſt to the ſecond is given; and if a given ſtraight line be taken from the ſecond, the ratio of the remainder to the third is given; alſo the rectangle contained by the firſt and third is given.

Let AB be the firſt ſtraight line, CD the ſecond, and EF the third. and becauſe the ratio of AB to CD is given, and that if a given ſtraight line be taken from CD, the ratio of the remainder to a. 24. Dat. EF is given; therefore [a] the exceſs of the firſt AB above a given ſtraight line has a given ratio to the third EF. Let BH be that given ſtraight line, therefore AH the exceſs of AB above it has a given ratio to EF; and b. 1. 6. conſequently [b] the rectangle BA, AH has a given ratio to the rectangle AB, EF, which laſt rectangle is given by the Hypotheſis; c. 2. Dat. therefore [c] the rectangle BA, AH is given, and BH the exceſs of its ſides is given; d. 85. Dat. wherefore the ſides AB, AH are given[d]. and becauſe the ratios of AB to CD, and of AH to EF are given; CD and EF are [c] given.



The Compoſition.

Let the given ratio of KL to KM be that which AB is required to have to CD; and let DG be the given ſtraight line which is to be taken from CD, and let the given ratio of KM to KN be that which the remainder muſt have to EF; alſo, let the given rectangle NK, KO be that to which the rectangle AB, EF is required to be equal. find the given ſtraight line BH which is to be taken from AB, which is done, as plainly appears from Prop. 24. Dat.

by making as KM to KL, ſo GD to HB. to the given ſtraight line BH apply [e] a rectangle equal to LK, KO exceeding by a ſquare, and let BA, AH be its ſides. then is AB the firſt of the ſtraight lines required to be found. and by making as LK to KM, ſo AB to DC, DC will be the ſecond. and laſtly, make as KM to KN, ſo CG to EF, and EF is the third. e. 29. 6.

For as AB to CD, ſo is HB to GD, each of theſe ratios being the ſame with the ratio of LK to KM; therefore [f] AH is to CG, as (AB to CD, that is, as) LK to KM; and as CG to EF, ſo is KM to KN; wherefore, ex aequali, as AH to EF, ſo is LK to KN. and as the rectangle BA, AH to the rectangle BA, EF, ſo is [g] the rectangle LK, KO to the rectangle KN, KO. and, by the Conſtruction, the rectangle BA, AH is equal to LK, KO, therefore [h] the rectangle AB, EF is equal to the given rectangle NK, KO. and AB has to CD the given ratio of KL to KM; and from CD the given ſtraight line GD being taken, the remainder CG has to EF the given ratio of KM to KN. Q. E. D. f. 19. 5. g. 1. 6. h. 14. 5.

PROB. II.

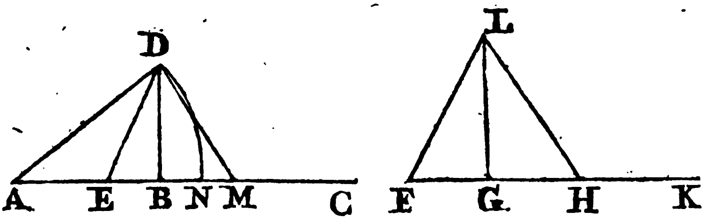
TO find three ſtraight lines ſuch, that the ratio of the firſt to the ſecond is given; and if a given ſtraight line be taken from the ſecond, the ratio of the remainder to the third is given; alſo the ſum of the ſquares of the firſt and third is given.

Let AB be the firſt ſtraight line, BC the ſecond, and BD the third. and becauſe the ratio of AB to BC is given, and that if a given ſtraight line be taken from BC, the ratio of the remainder to BD is given; therefore [a] the exceſs of the firſt AB above a given ſtraight line has a given ratio to the third BD. let AE be that given ſtraight line, therefore the remainder EB has a given ratio to BD. let BD be placed at right angles to EB, and join DE, then the triangle EBD is [b] given in ſpecies; wherefore the angle BED is given. let AE which is given in magnitude be given alſo in poſition, and the ſtraight line ED will be given [c] in poſition. join AD, and becauſe the ſum of the ſquares of AB, BD, that is [d], the ſquare of AD is given, therefore the ſtraight line AD is given in magnitude; and it is alſo given [e] in poſition, becauſe from the given point A it is drawn to the ſtraight line ED given in poſition. therefore the point D in which the two ſtraight lines AD, ED given in a. 24. Dat. b. 44. Dat. c. 32. Dat. d. 47. 1. e. 34. Dat.

f. 28. Dat poſition cut one another is given [f]. and the ſtraight line DB which
g. 33 Dat is at right angles to AB is given [g] in poſition, and AB is given in
poſition, therefore [f] the point B is given. and the points A, D are
h. 29. Dat. given, wherefore [h] the ſtraight lines AB, BD are given. and the
i. 2. Dat. ratio of AB to BC is given, and therefore [i] BC is given.

The Compoſition.

Let the given ratio of FG to GH be that which AB is required to have to BC, and let HK be the given ſtraight line which is to be taken from BC, and let the ratio which the remainder is required to have to BD be the given ratio of HG to GL, and place GL at right angles to FH, and join LF, LH. Next, as HG is to GF, ſo make HK to AE; produce AE to N ſo that AN be the ſtraight line to the ſquare of which the ſum of the ſquares of AB, BD is required to be equal; and make the angle NED equal to the angle GFL. from the center A at the diſtance AN deſcribe a circle, and let its circumference meet ED in D, and draw DB perpendicular to AN, and DM making the angle BDM equal to the angle GLH. laſtly, produce BM to C ſo that MC be equal to HK. then is AB the firſt, BC the ſecond and BD the third of the ſtraight lines that were to be found.



For the triangles EBD, FGL, as alſo DBM, LGH being equiangular, as EB to BD, ſo is FG to GL; and as DB to BM, ſo is LG to GH; therefore, ex aequali, as EB to BM, ſo is (FG to
k. 12. 5. GH, and ſo is) AE to HK or MC; wherefore [k] AB is to BC, as
AE to HK, that is, as FG to GH, that is, in the given ratio. and
from the ſtraight line BC taking MC which is equal to the given
ſtraight line HK, the remainder BM has to BD the given ratio of
d. 47. 1. HG to GL. and the ſum of the ſquares of AB, BD is equal [d] to
the ſquare of AD or AN which is the given ſpace. Q. E. D.

I believe it would be in vain to try to deduce the preceding Conſtruction from an Algebraical Solution of the Problem.

FINIS.

THE

ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY.

GLASGOW:
PRINTED BY J. & M. ROBERTSON,
SALTMARKET.
1803.

PLANE TRIGONOMETRY.

LEMMA I. Fig. 1.

LET ABC be a rectilineal angle, if about the point B as a center, and with any diſtance BA, a circle be deſcribed, meeting BA, BC, the ſtraight lines including the angle ABC in A, C; the angle ABC will be to four right angles, as the arch AC to the whole circumference.

Produce AB till it meet the circle again in F, and through B draw DE perpendicular to AB, meeting the circle in D, E.

By 33. 6. Elem. the angle ABC is to a right angle ABD, as the arch AC to the arch AD; and quadrupling the conſequents, the angle ABC will be to four right angles, as the arch AC to four times the arch AD, or to the whole circumference.

LEMMA II. Fig. 2.

LET ABC be a plane rectilineal angle as before; about B as a center with any two diſtances BD, BA, let two circles be deſcribed meeting BA, BC in D, E, A, C; the arch AC will be to the whole circumference of which it is an arch, as the arch DE is to the whole circumference of which it is an arch.

By Lemma 1. the arch AC is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; and by the ſame Lemma 1. the arch DE is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; therefore the arch AC is to the whole circumference of which it is an arch, as the arch DE to the whole circumference of which it is an arch.

DEFINITIONS. Fig. 3.

I.

LET ABC be a plane rectilineal angle; if about B as a center, with BA any diſtance, a circle ACF be deſcribed meeting BA, BC, in A, C; the arch AC is called the meaſure of the angle ABC.

II.

The circumference of a circle is ſuppoſed to be divided into 360 equal parts called degrees, and each degree into 60 equal parts called minutes, and each minute into 60 equal parts called ſeconds, &c. And as many degrees, minutes, ſeconds, &c. as are contained in any arch, of ſo many degrees, minutes, ſeconds, &c. is the angle, of which that arch is the meaſure, ſaid to be.

Cor. Whatever be the radius of the circle of which the meaſure of a given angle is an arch, that arch will contain the ſame number of degrees, minutes, ſeconds, &c. as is manifeſt from Lemma 2.

III.

Let AB be produced till it meet the circle again in F, the angle CBF, which, together with ABC, is equal to two right angles, is called the *Supplement* of the angle ABC.

IV.

A ſtraight line CD drawn through C, one of the extremities of the arch AC, perpendicular upon the diameter paſſing through the other extremity A, is called the *Sine* of the arch AC, or of the angle ABC, of which it is the meaſure.

Cor. The *Sine* of a quadrant, or of a right angle, is equal to the radius.

V.

The ſegment DA of the diameter paſſing through A, one extremity of the arch AC between the ſine CD, and that extremity is called the *Verſed Sine* of the arch AC, or angle ABC.

VI.

A ſtraight line AE touching the circle at A, one extremity of the arch AC, and meeting the diameter BC paſſing through the other extremity C in E, is called the *Tangent* of the arch AC, or of the angle ABC.

VII.

The ſtraight line BE between the center and the extremity of the tangent AE, is called the *Secant* of the arch AC, or angle ABC.

Cor. to Def. 4. 6. 7. The ſine, tangent, and ſecant of any angle ABC, are likewiſe the ſine, tangent, and ſecant of its ſupplement CBF.

It is manifeſt from Def. 4. that CD is the ſine of the angle CBF. Let CB be produced till it meet the circle again in G; and it is manifeſt that AE is the tangent, and BE the ſecant, of the angle ABG or EBF, from Def. 6. 7.

Fig. 4. Cor. to Def. 4. 5. 6. 7. The ſine, verſed ſine, tangent, and ſecant, of any arch which is the meaſure of any given angle ABC, is to the ſine, verſed ſine, tangent, and ſecant, of any

other arch which is the meaſure of the ſame angle, as the radius of the firſt is to the radius of the ſecond.

Let AC, MN be meaſures of the angles ABC, according to Def. 1. CD the ſine, DA the verſed ſine, AE the tangent, and BE the ſecant of the arch AC, according to Def. 4. 5. 6. 7. and NO the ſine, OM the verſed ſine, MP the tangent, and BP the ſecant of the arch MN, according to the ſame definitions. Since CD, NO, AE, MP are parallel, CD is to NO as the radius CB to the radius NB, and AE, to MP as AB to BM, and BC or BA to BD as BN or BM to BO; and, by converſion, DA to MO as AB to MB. Hence the corollary is manifeſt; therefore, if the radius be ſuppoſed to be divided into any given number of equal parts, the ſine, verſed ſine, tangent, and ſecant of any given angle, will each contain a given number of theſe parts; and, by trigonometrical tables, the length of the ſine, verſed ſine, tangent, and ſecant of any angle may be found in parts of which the radius contains a given number: and, vice verſa, a number expreſſing the length of the ſine, verſed ſine, tangent, and ſecant being given, the angle of which it is the ſine, verſed ſine, tangent, and ſecant may be found.

VIII.

Fig. 3.

The difference of an angle from a right angle is called the *Complement* of that angle. Thus, if BH be drawn perpendicular to AB, the angle CBH will be the complement of the angle ABC, or of CBF.

IX.

Let HK be the tangent, CL or DB, which is equal to it, the ſine, and BK the ſecant of CBH, the complement of ABC, according to Def. 4. 6. 7, HK is called the *co-tangent*, BD the *co-ſine*, and BK the *co-ſecant* of the angle ABC.

Cor. 1. The radius is a mean proportional between the tangent and co-tangent.

For, ſince HK, BA are parallel, the angles HKB, ABC will be equal, and the angles KHB, BAE are right; therefore the triangles BAE, KHB are ſimilar, and therefore AE is to AB, as BH or BA to HK.

Cor. 2. The radius is a mean proportional between the co-ſine and ſecant of any angle ABC.

Since CD, AE are parallel, BD is to BC or BA, as BA to BE.

PROP. I. Fig. 5.

IN a right angled plain triangle, if the hypothenuſe be made radius, the ſides become the ſines of the angles oppoſite to them; and if either ſide be made radius, the remaining ſide is the tangent of the angle oppoſite to it, and the hypothenuſe the ſecant of the ſame angle.

Let ABC be a right angled triangle; if the hypothenuſe BC be made radius, either of the ſides AC will be the ſine of the angle ABC oppoſite to it; and if either ſide BA be made radius, the other ſide AC will be the tangent of the angle ABC oppoſite to it, and the hypothenuſe BC the ſecant of the ſame angle.

About B as a center, with BC, BA for diſtances, let two circles CD, EA be deſcribed, meeting BA, BC in D, E: ſince CAB is a right angle, BC being radius, AC is the ſine of the angle ABC by def. 4. and BA being radius, AC is the tangent, and BC the ſecant of the angle ABC, by def. 6. 7.

Cor. 1. Of the hypothenuſe a ſide and an angle of a right angled triangle, any two being given, the third is alſo given.

Cor. 2. Of the two ſides and an angle of a right angled triangle, any two being given, the third is alſo given.

PROP. II. Fig. 6. 7.

THE ſides of a plane triangle are to one another, as the ſines of the angles oppoſite to them.

In right angled triangles this prop. is manifeſt from prop. 1. for if the hypothenuſe be made radius, the ſides are the ſines of the angles oppoſite to them, and the radius is the ſine of a right angle (cor. to def. 4.) which is oppoſite to the hypothenuſe.

In any oblique angled triangle ABC, any two ſides AB, AC will be to one another as the ſines of the angles ACB, ABC which are oppoſite to them.

From C, B draw CE, BD perpendicular upon the oppoſite ſides AB, AC produced, if need be. Since CEB, CDB are right angles, BC being radius, CE is the ſine of the angle CBA, and BD the ſine of the angle ACB; but the two triangles CAE, DAB have each a right angle at D and E; and likewiſe the common angle CAB; therefore they are ſimilar, and conſequently, CA is to AB, as CE to DB; that is, the ſides are as the ſines of the angles oppoſite to them.

Cor. Hence of two ſides, and two angles oppoſite to them, in a plain triangle, any three being given, the fourth is alſo given.

PROP. III. Fig. 8.

IN a plain triangle, the ſum of any two ſides is to their difference, as the tangent of half the ſum of the angles at the baſe, to the tangent of half their difference.

Let ABC be a plain triangle, the ſum of any two ſides AB, AC will be to their difference as the tangent of half the ſum of the angles at the baſe ABC, ACB to the tangent of half their difference.

About A as a center, with AB the greater ſide for a diſtance, let a circle be deſcribed, meeting AC produced in E, F, and BC in D; join DA, EB, FB; and draw FG parallel to BC, meeting EB in G.

The angle EAB (32. 1.) is equal to the ſum of the angles at the baſe, and the angle EFB at the circumference is equal to the half of EAB at the center (20. 3.); therefore EFB is half the ſum of the angles at the baſe; but the angle ACB (32. 1.) is equal to the angles CAD and ADC, or ABC together; therefore FAD is the difference of the angles at the baſe, and FBD at the circumference, or BFG, on account of the parallels FG, BD, is the half of that difference; but ſince the angle EBF in a ſemicircle is a right angle (1. of this) FB being radius, BE, BG, are the tangents of the angles EFB, BFG; but it is manifeſt that EC is the ſum of the ſides BA, AC, and CF their difference; and ſince BC, FG are parallel (2. 6.) EC is to CF, as EB to BG; that is, the ſum of the ſides is to their difference, as the tangent of half the ſum of the angles at the baſe to the tangent of half their difference.

PROP. IV. Fig. 18.

IN any plane triangle BAC, whoſe two ſides are BA, AC and baſe BC, the leſs of the two ſides, which let be BA, is to the greater AC as the radius is to the tangent of an angle; and the radius is to the tangent of the exceſs of this angle above half a right angle as the tangent of half the ſum of the angles B and C at the baſe, is to the tangent of half their difference.

At the point A, draw the ſtraight line EAD perpendicular to BA; make AE, AF, each equal to AB, and AD to AC; join BE, BF, BD, and from D, draw DG perpendicular upon BF. And becauſe BA is at right angles to EF, and EA, AB, AF are equal, each of the angles EBA, ABF is half a right angle, and the whole EBF is a right angle; alſo (4. 1. El.) EB is equal to BF. And ſince EBF, FGD are right angles, EB is parallel to GD, and the triangles EBF, FGD are ſimilar; therefore EB is to BF as DG to GF, and EB being equal to BF, FG muſt be equal to GD. And becauſe BAD is a right angle, BA the leſs ſide is to AD or AC the greater, as the radius is to the tangent of the angle ABD; and becauſe BGD is a right angle, BG is to GD or GF as the radius is to the tangent of GBD, which is the exceſs of the angle ABD above ABF half a right angle. But becauſe EB is parallel to GD, BG is to GF as ED is to DF, that is, ſince ED is the ſum of the ſides BA, AC and FD their difference, (3. of this,) as the tangent of half the ſum of the angles B, C, at the baſe to the tangent of half their difference. Therefore, in any plain triangle, &c. Q. E. D.

PROP. V. FIG. 9. and 10.

IN any triangle, twice the rectangle contained by any two ſides is to the difference of the ſum of the ſquares of theſe two ſides, and the ſquare of the baſe, as the radius is to the co-ſine of the angle included by the two ſides.

Let ABC be a plain triangle, twice the rectangle ABC contained by any two ſides BA, BC is to the difference of the ſum of the ſquares of BA, BC, and the ſquare of the baſe AC, as the radius to the co-ſine of the angle ABC.

From A, draw AD perpendicular upon the oppoſite ſide BC; then (by 12. and 13. 2. El.) the difference of the ſum of the ſquares of AB, BC, and the ſquare of the baſe AC, is equal to twice the rectangle CBD; but twice the rectangle CBA is to twice the rectangle CBD, that is, to the difference of the ſum of the ſquares of AB, BC, and the ſquare of AC, (1. 6.) as AB to BD; that is, by prop. 1. as radius to the ſine of BAD, which is the complement of the angle ABC, that is, as radius to the co-ſine of ABC.

PROP. VI. FIG. 11.

IN any triangle ABC, whoſe two ſides are AB, AC, and baſe BC; the rectangle contained by half the perimeter, and the exceſs of it above the baſe BC, is to the rectangle contained by the ſtraight lines, by which the half of the perimeter exceeds the other two ſides AB, AC, as the ſquare of the radius is to the ſquare of the tangent of half the angle BAC oppoſite to the baſe.

Let the angles BAC, ABC be biſected by the ſtraight lines AG, BG; and, producing the ſide AB, let the exterior angle CBH be biſected by the ſtraight line BK, meeting AG in K; and from the points G, K, let there be drawn perpendicular upon the ſides the ſtraight lines GD, GE, GF, KH, KL, KM. Since therefore (4. 4.) G is the center of the circle inſcribed in the triangle ABC, GD, GF, GE will be equal, and AD will be equal to AE, BD to BF, and CE to CF. In like manner KH, KL, KM will be equal, and BH will be equal to BM, and AH to AL, becauſe the angles HBM, HAL are biſected by the ſtraight lines BK, KA: and becauſe in the triangles KCL, KCM, the ſides LK, KM are equal, KC is common and KLC, KMC are right angles, CL will be equal to CM: ſince therefore BM is equal to BH, and CM to CL; BC will be equal to BH and CL together; and, adding AB and AC together, AB, AC, and BC will together be equal to AH and AL together: but AH, AL are equal: wherefore each of them is equal to half the perimeter of the triangle ABC: but ſince AD, AE are equal, and BD, BF, and alſo CE, CF, AB together with FC, will be equal to half the perimeter of the triangle to which AH or AL was ſhewn to be equal; taking away therefore the common AB, the remainder FC will be equal to the remainder BH: in the ſame manner is it demonſtrated, that BF is equal to CL: and ſince the points B, D, G, F, are in a circle, the angle DGF will be equal to the exterior and oppoſite angle FBH, (22. 3.); wherefore their halves BGD, HBK will be equal to one another; the right angled triangles BGD, HBK will therefore be equiangular, and GD will be to BD, as BH to HK, and the rectangle contained by GD, HK will be equal to the rectangle DBH or BFC: but ſince AH is to HK, as AD to DG, the rectangle HAD (22. 6.) will be to the rectangle contained by HK, DG, or the rectangle BFC, (as the ſquare of AD is to the ſquare of DG, that is) as the ſquare of the radius to the ſquare of the tangent of the angle DAG, that is, the half of BAC: but HA is half the perimeter of the triangle ABC, and AD is the exceſs of the ſame above HD, that is, above the baſe BC; but BF or CL is the exceſs of HA or AL above the ſide AC, and FC, or HB, is the exceſs of the ſame HA above the ſide AB; therefore the rectangle contained by half the perimeter, and the exceſs of the ſame above the baſe, viz. the rectangle HAD, is to the rectangle contained by the ſtraight lines by which the half of the perimeter exceeds the other two ſides, that is, the rectangle BFC, as the ſquare of the radius is to the ſquare of the tangent of half the angle BAC oppoſite to the baſe. Q. E. D.

PROP. VII. FIG. 12. 13.

IN a plain triangle, the baſe is to the ſum of the ſides, as the difference of the ſides is to the ſum or difference of the ſegments of the baſe made by the perpendicular upon it from the vertex, according as the ſquare of the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide and the baſe.

Let ABC be a plane triangle; if from A the vertex be drawn a ſtraight line AD perpendicular upon the baſe BC, the baſe BC will be to the ſum of the ſides BA, AC, as the difference of the ſame ſides is to the ſum or difference of the ſegments CD, BD, according as the ſquare of AC the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide AB, and the baſe BC.

About A as a center, with AC the greater ſide for a diſtance, let a circle be deſcribed meeting AB produced in E, F, and CB in G: it is manifeſt that FB is the ſum, and BE the difference of the ſides; and ſince AD is perpendicular to GC, GD, CD will be equal; conſequently GB will be equal to the ſum or difference of the ſegments CD, BD, according as the perpendicular AD meets the baſe, or the baſe produced; that is, (by Conv. 12. and 13. 2.) according as the ſquare of AC is greater or leſs than the ſum of the ſquares of AB, BC: but (by 35. 3.) the rectangle CBG is equal to the rectangle EBF: that is, (16. 6.) BC is to BF, as BE is to BG; that is, the baſe is to the ſum of the ſides, as the difference of the ſides is to the ſum or difference of the ſegments of the baſe made by the perpendicular from the vertex, according as the ſquare of the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide and the baſe. Q. E. D.

PROP. VIII. PROB. FIG. 14.

THE ſum and difference of two magnitudes being given, to find them.

Half the given ſum added to half the given difference, will be the greater, and half the difference ſubtracted from half the ſum, will be the leſs.

For, let AB be the given ſum, AC the greater, and BC the leſs. Let AD be half the given ſum; and to AD, DB, which are equal, let DC be added, then AC will be equal to BD, and DC together; that is, to BC, and twice DC; conſequently twice DC is the difference, and DC half that difference; but AC the greater is equal to AD, DC; that is, to half the ſum added to half the difference, and BC the leſs is equal to the exceſs of BD, half the ſum above DC half the difference. Q. E. D.

SCHOLIUM.

Of the ſix parts of a plane triangle (the three ſides and three angles) any three being given, to find the other three is the buſineſs of plane trigonometry; and the ſeveral caſes of that problem may be reſolved by means of the preceding propoſitions, as in the two following, with the tables annexed. In theſe, the ſolution is expreſſed by a fourth proportional to three given lines; but if the given parts be expreſſed by numbers from trigonometrical tables, it may be obtained arithmetically by the common Rule of Three.

Note. In the Tables the following abbreviations are uſed. R, is put for the Radius; T, for Tangent; and S, for Sine. Degrees, minutes, ſeconds, &c. are written in this manner; 30° 25′ 13″, &c. which ſignifies 30 degrees, 25 minutes, 13 ſeconds, &c.

SOLUTION of the CASES of Right Angled TRIANGLES.

GENERAL PROPOSITION.

IN a right angled triangle, of the three ſides and three angles, any two being given beſides the right angle, the other three may be found, except when the two acute angles are given, in which caſe the ratios of the ſides are only given, being the ſame with the ratios of the ſines of the angles oppoſite to them.

It is manifeſt from 47. 1. that of the two ſides and hypothenuſe any two be given the third may alſo be found. It is alſo manifeſt from 32. 1. that if one of the acute angles of a right-angled triangle be given, the other is alſo given, for it is the complement of the former to a right angle.

If two angles of any triangle be given, the third is alſo given, being the ſupplement of the two given angles to two right angles.

Fig. 15. The other caſes may be reſolved by help of the preceding propoſitions, as in the following table.

	GIVEN.	SOUGHT.	
1	Two ſides, AB AC.	The angles B, C.	AB : AC : : R : T, B, of which C is the complement.
2	AB, BC, a ſide and the hypothenuſe.	The angles B, C.	BC : BA : : R : S, C, of which B is the complement.
3	AB, B, a ſide and an angle.	The other ſide AC.	R : T, B : : BA : AC.
4	AB and B, a ſide and an angle.	The hypothenuſe BC.	S, C : R : : BA : BC.
5	BC and B, the hypothenuſe and an angle.	The ſide AC.	R : S, B : : BC : CA.

Theſe five caſes are reſolved by prop. 1.

SOLUTION of the CASES of Oblique-Angled TRIANGLES.

GENERAL PROPOSITION.

IN an oblique-angled triangle, of the three ſides and three angles, any three being given, the other three may be found, except when the three angles are given; in which caſe the ratios of the ſides are only given, being the ſame with the ratios of the ſines of the angles oppoſite to them.

	GIVEN.	SOUGHT.		
1	A, B, and therefore C, and the ſide AB.	BC, AC.	S, C : S, A : : AB : BC, and alſo S, C : S, B : : AB : AC. (2.)	Fig. 16. 17.
2	AB, AC, and B, two ſides and an angle oppoſite to one of them.	The angles A and C.	AC : AB : : S, B : S, C. (2.) This caſe admits of two ſolutions; for C may be greater or leſs than a quadrant. (Cor. to def. 4.)	
3	AB, AC, and A, two ſides, and the included angle	The angles B and C.	AB+AC : AB—AC : : T, $\frac{C+B}{2}$: T, $\frac{C-B}{2}$: (3.) and the ſum and difference of the angles C, B, being given, each of them is given. (7.) Otherwiſe. FIG. 18. BA : AC : : R : T, ABC, and alſo R : T, ABC—45° : T, $\frac{B+C}{2}$: T, $\frac{B-C}{2}$: (4.) therefore B and C are given as before. (7.)	

	GIVEN.	SOUGHT.	
4	AB, BC, CA, the three ſides.	A, B, C, the three angles.	$2\,AC\times CB : ACq+CBq-ABq :: R : CoS, C$. If $ACq+CBq$ be greater than ABq. FIG. 16. $2\,AC\times CB : ABq-ACq-CBq :: R : CoS, C$. If ABq be greater than $ACq+CBq$ FIG. 17. (4.) Otherwiſe. Let $AB+BC+AC=2\,P$. $P\times\overline{P-AB} : \overline{P-AC}\times\overline{P-BC} :: Rq : Tq, \frac{1}{2}C$, and hence C is known. (5.) Otherwiſe. Let AD be perpendicular to BC. 1. If ABq be leſs than $ACq+CBq$. Fig. 16. $BC : BA+AC :: BA-AC : BD-DC$, and BC the ſum of BD, DC is given; therefore each of them is given. (7.) 2 If ABq be greater than $ACq+CBq$. Fig. 17. $BC : BA+AC :: BA-AC : BD+DC$; and BC the difference of BD, DC is given, therefore each of them is given. (7.) And $CA : CD :: R : CoS, C$. (1.) and C being found, A and B are found by caſe 2. or 3.

SPHERICAL TRIGONOMETRY.

DEFINITIONS.

I.

THE pole of a circle of the ſphere is a point in the ſuperficies of the ſphere, from which all ſtraight lines drawn to the circumference of the circle are equal.

II.

A great circle of the ſphere is any whoſe plane paſſes through the center of the ſphere, and whoſe center therefore is the ſame with that of the ſphere.

III.

A ſpherical triangle is a figure upon the ſuperficies of a ſphere comprehended by three arches of three great circles, each of which is leſs than a ſemicircle.

IV.

A ſpherical angle is that which on the ſuperficies of a ſphere is contained by two arches of great circles, and is the ſame with the inclination of the planes of theſe great circles.

PROP. I.

GREAT circles biſect one another.

As they have a common center their common ſection will be a diameter of each which will biſect them.

PROP. II. FIG. 1.

THE arch of a great circle betwixt the pole and circumference of another is a quadrant.

Let ABC be a great circle and D its pole; if a great circle DC paſs through D, and meet ABC in C, the arch DC will be a quadrant.

Let the great circle CD meet ABC again in A, and let AC be the common ſection of the great circles, which will paſs through E the center of the ſphere: join DE, DA, DC: by def. 1. DA, DC are equal, and AE, EC are alſo equal, and DE is common; therefore (8. 1.) the angles DEA, DEC are equal; wherefore the arches DA, DC are equal, and conſequently each of them is a quadrant. Q. E. D.

PROP. III. Fig. 2.

If a great circle be deſcribed meeting two great circles AB, AC paſſing through its pole A in B, C, the angle at the center of the ſphere upon the circumference BC, is the ſame with the ſpherical angle BAC, and the arch BC is called the meaſure of the ſpherical angle BAC.

Let the planes of the great circles AB, AC interſect one another in the ſtraight line AD paſſing through D their common center; join DB, DC.

Since A is the pole of BC, AB, AC will be quadrants, and the angles ADB, ADC right angles; therefore (6. def. 11.) the angle CDB is the inclination of the planes of the circles AB, AC; that is, (def. 4.) the ſpherical angle BAC. Q. E. D.

Cor. If through the point A, two quadrants AB, AC, be drawn, the point A will be the pole of the great circle BC, paſſing through their extremities B, C.

Join AC, and draw AE a ſtraight line to any other point E in BC; join DE: ſince AC, AB are quadrants, the angles ADB, ADC are right angles, and AD will be perpendicular to the plane of BC: therefore the angle ADE is a right angle, and AD, DC are equal to AD, DE each to each; therefore AE, AC are equal, and A is the pole of BC, by def. 1. Q. E. D.

PROP. IV. Fig. 3.

In iſoſceles ſpherical triangles, the angles at the baſe are equal.

Let ABC be an iſoſceles triangle, and AC, CB the equal ſides; the angles BAC, ABC, at the baſe AB, are equal.

Let D be the center of the ſphere, and join DA, DB, DC; in DA take any point E, from which draw, in the plane ADC, the ſtraight line EF at right angles to ED meeting CD in F, and draw, in the plane ADB, EG at right angles to the ſame ED; therefore the rectilineal angle FEG is (6. def. 11.) the inclination of the planes ADC, ADB, and therefore is the ſame with the ſpherical angle BAC: from F draw FH perpendicular to DB, and from H draw, in the plane ADB, the ſtraight line HG at right angles to HD meeting EG in G, and join GF. Becauſe DE is at right angles to EF and EG, it is perpendicular to the plane FEG, (4. 11.) and therefore the plane FEG is perpendicular to the plane ADB, in which DE is: (18. 11.) in the ſame manner the plane FHG is perpendicular to the plane ADB; and therefore GF the common ſection of the planes FEG, FHG is perpendicular to the plane ADB; (19. 11.) and becauſe the angle FHG is the inclination of the planes BDC, BDA, it is the ſame with the ſpherical angle ABC; and the ſides AC, CB of the ſpherical triangle being equal, the angles EDF, HDF, which ſtand upon them at the center of the ſphere, are equal; and in the triangles EDF, HDF the ſide DF is common, and the angles DEF, DHF are right angles; therefore EF, FH are equal; and in the triangles FEG, FHG the ſide GF is common and the ſides EG, GH will be equal by the 47. 1. and therefore the angle FEG is equal to FHG; (8. 1.) that is, the ſpherical angle BAC is equal to the ſpherical angle ABC.

PROP. V. FIG. 3.

IF, in a ſpherical triangle ABC, two of the angles BAC, ABC be equal, the ſides BC, AC oppoſite to them, are equal.

Read the conſtruction and demonſtration of the preceding propoſition, unto the words, " and the ſides AC, CB," &c. and the reſt of the demonſtration will be as follows, viz.

And the ſpherical angles BAC, ABC being equal, the rectilineal angles FEG, FHG, which are the ſame with them, are equal; and in the triangles FGE, FGH the angles at G are right angles, and the ſide FG oppoſite to two of the equal angles is common; therefore (26. 1.) EF is equal to FH; and in the right-angled triangles DEF, DHF the ſide DF is common; wherefore (47. 1.) ED is equal to DH, and the angles EDF, HDF are therefore equal, (4. 1.) and conſequently the ſides AC, BC of the ſpherical triangle are equal.

PROP. VI. Fig. 4.

ANY two ſides of a ſpherical triangle are greater than the third.

Let ABC be a ſpherical triangle, any two ſides AB, BC will be greater than the other ſide AC.

Let D be the center of the ſphere; join DA, DB, DC.

The ſolid angle at D is contained by three plane angles ADB, ADC, BDC; and by 20. 11. any two of them ADB, BDC are greater than the third ADC; that is, any two ſides AB, BC of the ſpherical triangle ABC, are greater than the third AC.

PROP. VII. Fig. 4.

THE three ſides of a ſpherical triangle are leſs than a circle.

Let ABC be a ſpherical triangle as before, the three ſides AB, BC, AC are leſs than a circle.

Let D be the center of the ſphere: the ſolid angle at D is contained by three plane angles BDA, BDC, ADC, which together are leſs than four right angles, (21. 11.) therefore the ſides AB, BC, AC together, will be leſs than four quadrants; that is, leſs than a circle.

PROP. VIII. Fig. 5.

IN a ſpherical triangle the greater angle is oppoſite to the greater ſide; and converſely.

Let ABC be a ſpherical triangle, the greater angle A is oppoſed to the greater ſide BC.

Let the angle BAD be made equal to the angle B, and then BD, DA will be equal, (5. of this) and therefore AD, DC are equal to BC; but AD, DC are greater than AC, (6. of this,) therefore BC is greater than AC, that is, the greater angle A is oppoſite to the greater ſide BC. The converſe is demonſtrated as prop. 19. 1. El. Q. E. D.

PROP. IX. Fig. 6.

IN any ſpherical triangle ABC, if the ſum of the ſides AB, BC be greater, equal, or leſs than a ſemicircle, the internal angle at the baſe AC will be greater, equal, or leſs than the external and oppoſite BCD; and therefore the ſum of the angles A and ACB will be greater, equal, or leſs than two right angles.

Let AC, AB produced meet in D.

1. If AB, BC be equal to a ſemicircle, that is, to AD, BC, BD will be equal, that is, (4. of this) the angle D, or the angle A will be equal to the angle BCD.

2. If AB, BC together be greater than a ſemicircle, that is, greater than ABD, BC will be greater than BD; and therefore (8. of this) the angle D, that is, the angle A, is greater than the angle BCD.

3. In the ſame manner is it ſhewn, that if AB, BC together be leſs than a ſemicircle, the angle A is leſs than the angle BCD. And ſince the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two right angles; and if A be leſs than BCD, A and ACB will be leſs than two right angles. Q. E. D.

PROP. X. Fig. 7.

IF the angular points A, B, C of the ſpherical triangle ABC be the poles of three great circles, theſe great circles by their interſections will form another triangle FDE, which is called ſupplemental to the former; that is, the ſides FD, DE, EF are the ſupplements of the meaſures of the oppoſite angles C, B, A, of the triangle ABC, and the meaſures of the angles F, D, E of the triangle FDE, will be the ſupplements of the ſides AC, BC, BA, in the triangle ABC.

Let AB produced meet DE, EF in G, M, and AC meet FD, FE in K, L, and BC meet FD, DE in N, H.

Since A is the pole of FE, and the circle AC paſſes through A, EF will paſs through the pole of AC, (13. 15. 1. Th.) and ſince AC paſſes through C, the pole of FD, FD will paſs through the pole of AC; therefore the pole of AC is in the point F, in which the arches DF, EF interſect each other. In the ſame manner, D is the pole of BC, and E the pole of AB.

And ſince F, E are the poles of AL, AM, FL and EM are quadrants, and FL, EM together, that is FE and ML together, are equal to a ſemicircle. But ſince A is the pole of ML, ML is the meaſure of the angle BAC, conſequently, FE is the ſupplement of the meaſure of the angle BAC. In the ſame manner, ED, DF are the ſupplements of the meaſures of the angles ABC, BCA.

Since likewiſe CN, BH are quadrants, CN, BH together, that is, NH, BC together, are equal to a ſemicircle; and ſince D is the pole of NH, NH is the meaſure of the angle FDE, therefore the meaſure of the angle FDE is the ſupplement of the ſide BC. In the ſame manner, it is ſhewn that the meaſures of the angles DEF, EFD are the ſupplements of the ſides AB, AC, in the triangle ABC. Q. E. D.

PROP. XI. Fig. 7.

THE three angles of a ſpherical triangle are greater than two right angles, and leſs than ſix right angles.

The meaſures of the angles A, B, C, in the triangle ABC, together with the three ſides of the ſupplemental triangle DEF, are (10. of this) equal to three ſemicircles; but the three ſides of the triangle FDE, are (7. of this) leſs than two ſemicircles; therefore the meaſures of the angles A, B, C are greater than a ſemicircle; and hence the angles A, B, C are greater than two right angles.

All the external and internal angles of any triangle are equal to ſix right angles; therefore, all the internal angles are leſs than ſix right angles.

PROP. XII. Fig. 8.

IF from any point C, which is not the pole of the great circle ABD, there be drawn arches of great circles CA, CD, CE, CF, &c. the greateſt of theſe is

CA, which paſſes through H the pole of ABD, and CB the remainder of ACB is the leaſt, and of any others CD, CE, CF, &c. CD, which is nearer to CA, is greater than CE, which is more remote.

Let the common ſection of the planes of the great circles ACB, ADB be AB; and from C, draw CG perpendicular to AB, which will alſo be perpendicular to the plane ADB; (4. def. 11.) join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the ſtraight lines drawn from G to the circumference ADB, GA is the greateſt, and GB the leaſt; (7. 3.) and GD which is nearer to GA is greater than GF, which is more remote. The triangles CGA, CGD are right-angled at G, and they have the common ſide CG; therefore the ſquares of CG, GA together, that is, the ſquare of CA, is greater than the ſquares of CG, GD together, that is, the ſquare of CD; and CA is greater than CD, and therefore the arch CA is greater than CD. In the ſame manner, ſince GD is greater than GE, and GE than GF, &c. it is ſhewn that CD is greater than CE, and CE than CF, &c. and conſequently, the arch CD greater than the arch CE, and the arch CE greater than the arch CF, &c. And ſince GA is the greateſt, and GB the leaſt of all the ſtraight lines drawn from G to the circumference ADB, it is manifeſt that CA is the greateſt, and CB the leaſt of all the ſtraight lines drawn from C to that circumference; and therefore the arch CA is the greateſt, and CB the leaſt of all the circles drawn through C, meeting ADB. Q. E. D.

PROP. XIII. FIG. 9.

IN a right-angled ſpherical triangle the ſides are of the ſame affection with the oppoſite angles; that is, if the ſides be greater or leſs than quadrants, the oppoſite angles will be greater or leſs than right angles.

Let ABC be a ſpherical triangle right-angled at A, any ſide AB, will be of the ſame affection with the oppoſite angle ACB.

Caſe 1. Let AB be leſs than a quadrant, let AE be a quadrant, and let EC be a great circle paſſing through E, C. Since A is a right angle, and AE a quadrant, E is the pole of the great circle AC, and ECA a right angle; but ECA is greater than BCA, therefore BCA is leſs than a right angle. Q. E. D.

Fig. 10. Caſe 2. Let AB be greater than a quadrant, make AE a quadrant, and let a great circle paſs through C, E. ECA is a right angle as before, and BCA is greater than ECA, that is, greater than a right angle. Q. E. D.

PROP. XIV.

If the two ſides of a right-angled ſpherical triangle be of the ſame affection, the hypothenuſe will be leſs than a quadrant; and if they be of different affection, the hypothenuſe will be greater than a quadrant.

Let ABC be a right-angled ſpherical triangle, if the two ſides AB, AC be of the ſame or of different affection, the hypothenuſe BC will be leſs or greater than a quadrant.

Fig. 9. Caſe 1. Let AB, AC be each leſs than a quadrant. Let AE, AG be quadrants; G will be the pole of AB, and E the pole of AC, and EC a quadrant; but, by prop. 12. CE is greater than CB, ſince CB is farther off from CGD than CE. In the ſame manner, it is ſhewn that CB, in the triangle CBD, where the two ſides CD, BD are each greater than a quadrant, is leſs than CE, that is, leſs than a quadrant. Q. E. D.

Fig. 10. Caſe 2. Let AC be leſs, and AB greater than a quadrant; then the hypothenuſe BC will be greater than a quadrant; for let AE be a quadrant, then E is the pole of AC, and EC will be a quadrant. But CB is greater than CE by prop. 12. ſince AC paſſes through the pole of ABD. Q. E. D.

PROP. XV.

If the hypothenuſe of a right-angled triangle be greater or leſs than a quadrant, the ſides will be of different or the ſame affection.

This is the converſe of the preceding, and demonſtrated in the ſame manner.

PROP. XVI.

IN any ſpherical triangle ABC, if the perpendicular AD from A upon the baſe BC fall within the triangle, the angles B and C at the baſe will be of the ſame affection; and if the perpendicular fall without the triangle, the angles B and C will be of different affection.

1. Let AD fall within the triangle; then (13. of this) ſince ADB, ADC are right-angled ſpherical triangles, the angles B, C muſt each be of the ſame affection as AD. Fig. 11.

2. Let AD fall without the triangle, then (13. of this) the angle B is of the ſame affection as AD; and by the ſame, the angle ACD is of the ſame affection as AD; therefore the angle ACB and AD are of different affection, and the angles B and ACB of different affection. Fig. 12.

Cor. Hence if the angles B and C be of the ſame affection, the perpendicular will fall within the baſe; for, if it did not, (16. of this,) B and C would be of different affection. And if the angles B and C be of oppoſite affection, the perpendicular will fall without the triangle; for, if it did not, (16. of this,) the angles B and C would be of the ſame affection, contrary to the ſuppoſition.

PROP. XVII. Fig. 13.

IN right-angled ſpherical triangles, the ſine of either of the ſides about the right angle, is to the radius of the ſphere, as the tangent of the remaining ſide is to the tangent of the angle oppoſite to that ſide.

Let ABC be a triangle, having the right angle at A; and let AB be either of the ſides, the ſine of the ſide AB will be to the radius, as the tangent of the other ſide AC to the tangent of the angle ABC, oppoſite to AC. Let D be the center of the ſphere; join AD, BD, CD, and let AE be drawn perpendicular to BD, which therefore will be the ſine of the arch AB, and from the point E, let there be drawn in the plane BDC the ſtraight line EF at right angles to BD, meeting DC in F, and let AF be joined. Since therefore the ſtraight line DE is at right angles to both EA and EF, it will alſo be at right angles to the plane AEF, (4. 11.) wherefore the plane ABD, which paſſes through DE, is perpendicular to the plane AEF, (18. 11.) and the plane AEF perpendicular to ABD: the plane ACD or AFD is alſo perpendicular to the ſame ABD: therefore the common ſection, viz. the ſtraight line AF, is at right angles to the plane ABD: (19. 11.) and FAE, FAD are right angles; (3. def. 11.) therefore AF is the tangent of the arch AC; and in the rectilineal triangle AEF, having a right angle at A, AE will be to the radius as AF to the tangent of the angle AEF, (1. Pl. Tr.;) but AE is the ſine of the arch AB, and AF the tangent of the arch AC, and the angle AEF is the inclination of the planes CBD, ABD, (6. def. 11.) or the ſpherical angle ABC; therefore the ſine of the arch AB is to the radius as the tangent of the arch AC, to the tangent of the oppoſite angle ABC.

Cor. 1. If therefore of the two ſides, and an angle oppoſite to one of them, any two be given, the third will alſo be given.

Cor. 2. And ſince by this propoſition the ſine of the ſide AB is to the radius, as the tangent of the other ſide AC to the tangent of the angle ABC oppoſite to that ſide; and as the radius is to the co-tangent of the angle ABC, ſo is the tangent of the ſame angle ABC to the radius, (Cor. 2. def. Pl. Tr.) by equality, the ſine of the ſide AB is to the co-tangent of the angle ABC adjacent to it, as the tangent of the other ſide AC to the radius.

PROP. XVIII. Fig. 13.

IN right-angled ſpherical triangles the ſine of the hypothenuſe is to the radius, as the ſine of either ſide is to the ſine of the angle oppoſite to that ſide.

Let the triangle ABC be right-angled at A, and let AC be either of the ſides; the ſine of the hypothenuſe BC will be to the radius as the ſine of the arch AC is to the ſine of the angle ABC.

Let D be the center of the ſphere, and let CG be drawn perpendicular to DB, which will therefore be the ſine of the hypothenuſe BC; and from the point G let there be drawn in the plane ABD the ſtraight line GH perpendicular to DB, and let CH be joined: CH will be at right angles to the plane ABD, as was ſhewn in the preceding propoſition of the ſtraight line FA: wherefore CHD, CHG are right angles, and CH is the ſine of the arch AC; and in the triangle CHG, having the right angle CHG, CG is to the radius as CH to the ſine of the angle CGH: (1. Pl. Tr.) but ſince CG, HG are at right angles to DGB, which is the common ſection of the planes CBD, ABD, the angle CGH will be equal to the inclination of theſe planes; (6. def. 11.) that is, to the ſpherical angle ABC. The ſine, therefore, of the hypothenuſe CB is to the radius as the ſine of the ſide AC is to the ſine of the oppoſite angle ABC. Q. E. D.

Cor. Of theſe three, viz. the hypothenuſe, a ſide, and the angle oppoſite to that ſide, any two being given, the third is alſo given by prop. 2.

PROP. XIX. Fig. 14.

In right-angled ſpherical triangles, the co-ſine of the hypothenuſe is to the radius as the co-tangent of either of the angles is to the tangent of the remaining angle.

Let ABC be a ſpherical triangle, having a right angle at A, the co-ſine of the hypothenuſe BC will be to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB.

Deſcribe the circle DE, of which B is the pole, and let it meet AC in F, and the circle BC in E; and ſince the circle BD paſſes through the pole B of the circle DF, DF will alſo paſs through the pole of BD. (13. 18. 1. Theod. ſph.) And ſince AC is perpendicular to BD, AC will alſo paſs through the pole of BD; wherefore the pole of the circle BD will be found in the point where the circles AC, DE meet, that is, in the point F: the arches FA, FD are therefore quadrants, and likewiſe the arches BD, BE: in the triangle CEF, right-angled at the point E, CE is the complement of the hypothenuſe BC of the triangle ABC, EF is the complement of the arch ED, which is the meaſure of the angle ABC, and FC the hypothenuſe of the triangle CEF, is the complement of AC, and the arch AD, which is the meaſure of the angle CFE, is the complement of AB.

But (17. of this) in the triangle CEF, the ſine of the ſide CE is to the radius, as the tangent of the other ſide is to the tangent of the angle ECF oppoſite to it, that is, in the triangle ABC, the co-ſine of the hypothenuſe BC is to the radius, as the co-tangent of the angle ABC is to the tangent of the angle ACB. Q. E. D.

Cor. 1. Of theſe three, viz. the hypothenuſe and the two angles, any two being given, the third will alſo be given.

Cor. 2. And ſince by this propoſition the co-ſine of the hypothenuſe BC is to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB. But as the radius is to the co-tangent of the angle ACB, ſo is the tangent of the ſame to the radius; (Cor. 2. def. Pl. Tr.) and, ex aequo, the co-ſine of the hypothenuſe BC is to the co-tangent of the angle ACB, as the co-tangent of the angle ABC to the radius.

PROP. XX. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of an angle is to the radius, as the tangent of the ſide adjacent to that angle is to the tangent of the hypothenuſe.

The ſame conſtruction remaining; in the triangle CEF. (17. of this) the ſine of the ſide EF is to the radius, as the tangent of the other ſide CE is to the tangent of the angle CFE oppoſite to it; that is, in the triangle ABC, the co-ſine of the angle ABC is to the radius as (the co-tangent of the hypothenuſe BC to the co-tangent of the ſide AB, adjacent to ABC, or as) the tangent of the ſide AB to the tangent of the hypothenuſe, ſince the tangents of two arches are reciprocally proportional to their co-tangents. (Cor. 1. def. Pl. Tr.)

Cor. And ſince by this propoſition the co-ſine of the angle ABC is to the radius, as the tangent of the ſide AB is to the tangent of the hypothenuſe BC; and as the radius is to the co-tangent of BC, ſo is the tangent of BC to the radius; by equality, the co-ſine of the angle ABC will be to the co-tangent of the hypothenuſe BC, as the tangent of the ſide AB, adjacent to the angle ABC, to the radius.

PROP. XXI. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the hypothenuſe is to the co-ſine of the other ſide.

The ſame conſtruction remaining; in the triangle CEF, the ſine of the hypothenuſe CF is to the radius, as the ſine of the ſide CE to the ſine of the oppoſite angle CFE; (18. of this) that is, in the triangle ABC the co-ſine of the ſide CA is to the radius as the co-ſine of the hypothenuſe BC to the co-ſine of the other ſide BA. Q. E. D.

PROP. XXII. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the angle oppoſite to that ſide is to the ſine of the other angle.

The ſame conſtruction remaining; in the triangle CEF, the ſine of the hypothenuſe CF is to the radius as the ſine of the ſide EF is to the ſine of the angle ECF oppoſite to it; that is, in the triangle ABC, the co-ſine of the ſide CA is to the radius, as the co-ſine of the angle ABC oppoſite to it, is to the ſine of the other angle. Q. E. D.

of the angle ECF oppoſite to it, that is, in the triangle ABC, the co-ſine of the hypothenuſe BC is to the radius, as the co-tangent of the angle ABC is to the tangent of the angle ACB. Q. E. D.

Cor. 1. Of theſe three, viz. the hypothenuſe and the two angles, any two being given, the third will alſo be given.

Cor. 2. And ſince by this propoſition the co-ſine of the hypothenuſe BC is to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB. But as the radius is to the co-tangent of the angle ACB, ſo is the tangent of the ſame to the radius; (Cor. 2. def. Pl. Tr.) and, ex aequo, the co-ſine of the hypothenuſe BC is to the co-tangent of the angle ACB, as the co-tangent of the angle ABC to the radius.

PROP. XX. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of an angle is to the radius, as the tangent of the ſide adjacent to that angle is to the tangent of the hypothenuſe.

The ſame conſtruction remaining; in the triangle CEF. (17. of this) the ſine of the ſide EF is to the radius, as the tangent of the other ſide CE is to the tangent of the angle CFE oppoſite to it; that is, in the triangle ABC, the co-ſine of the angle ABC is to the radius as (the co-tangent of the hypothenuſe BC to the co-tangent of the ſide AB, adjacent to ABC, or as) the tangent of the ſide AB to the tangent of the hypothenuſe, ſince the tangents of two arches are reciprocally proportional to their co-tangents. (Cor. 1. def. Pl. Tr.)

Cor. And ſince by this propoſition the co-ſine of the angle ABC is to the radius, as the tangent of the ſide AB is to the tangent of the hypothenuſe BC; and as the radius is to the co-tangent of BC, ſo is the tangent of BC to the radius; by equality, the co-ſine of the angle ABC will be to the co-tangent of the hypothenuſe BC, as the tangent of the ſide AB, adjacent to the angle ABC, to the radius.

PROP. XXI. FIG. 14.

IN right-angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the hypothenuſe is to the co-ſine of the other ſide.

The ſame conſtruction remaining; in the triangle CEF, the ſine of the hypothenuſe CF is to the radius, as the ſine of the ſide CE to the ſine of the oppoſite angle CFE; (18. of this) that is, in the triangle ABC the co-ſine of the ſide CA is to the radius as the co-ſine of the hypothenuſe BC to the co-ſine of the other ſide BA. Q. E. D.

PROP. XXII. FIG. 14.

IN right-angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the angle oppoſite to that ſide is to the ſine of the other angle.

The ſame conſtruction remaining; in the triangle CEF, the ſine of the hypothenuſe CF is to the radius as the ſine of the ſide EF is to the ſine of the angle ECF oppoſite to it; that is, in the triangle ABC, the co-ſine of the ſide CA is to the radius, as the co-ſine of the angle ABC oppoſite to it, is to the ſine of the other angle. Q. E. D.

OF THE CIRCULAR PARTS.

Fig. 15. IN any right-angled ſpherical triangle ABC, the complement of the hypothenuſe, the complements of the angles, and the two ſides, are called *The circular parts of the triangle*, as if it were following each other in a circular order, from whatever part we begin: thus, if we begin at the complement of the hypothenuſe, and proceed towards the ſide BA, the parts following in order will be the complement of the hypothenuſe, the complement of the angle B, the ſide BA, the ſide AC, (for the right angle at A is not reckoned among the parts,) and, laſtly, the complement of the angle C. And thus at whatever part we begin, if any three of theſe five be taken, they either will be all contiguous or adjacent, or one of them will not be contiguous to either of the other two: in the firſt caſe, the part which is between the other two is called the *Middle part*, and the other two are called *Adjacent extremes*. In the ſecond caſe, the part which is not contiguous to either of the other two is called the *Middle part*, and the other two *Oppoſite extremes*. For example, if the three parts be the complement of the hypothenuſe BC, the complement of the angle B, and the ſide BA; ſince theſe three are contiguous to each other, the complement of the angle B will be the middle part, and the complement of the hypothenuſe BC and the ſide BA will be adjacent extremes: but if the complement of the hypothenuſe BC, and the ſides BA, AC be taken; ſince the complement of the hypothenuſe is not adjacent to either of the ſides, viz. on account of the complements of the two angles B and C interveening between it and the ſides, the complement of the hypothenuſe BC will be the middle part, and the ſides, BA, AC oppoſite extremes. The moſt acute and ingenious Baron Napier, the inventor of Logarithms, contrived the two following rules concerning theſe parts, by means of which all the caſes of right-angled ſpherical triangles are reſolved with the greateſt eaſe.

RULE I.

The rectangle contained by the radius and the ſine of the middle part, is equal to the rectangle contained by the tangents of the adjacent parts.

RULE II.

The rectangle contained by the radius, and the ſine of the middle part is equal to the rectangle contained by the co-ſines of the oppoſite parts.

Theſe rules are demonſtrated in the following manner.

Firſt, Let either of the ſides, as BA, be the middle part, and therefore the complement of the angle B, and the ſide AC will be adjacent extremes. And by Cor. 2. prop. 17. of this, S, BA is to the Co-T, B, as T, AC is to the radius, and therefore R×S, BA=Co-T, B×T, AC. Fig. 16.

The ſame ſide BA being the middle part, the complement of the hypothenuſe, and the complement of the angle C, are oppoſite extremes; and by prop. 18. S, BC is to the radius, as S, BA to S, C; therefore R×S, BA=S, BC×S, C.

Secondly, Let the complement of one of the angles, as B, be the middle part, and the complement of the hypothenuſe, and the ſide BA will be adjacent extremes: and by Cor. prop. 20. Co-S, B is to Co-T, BC, as T, BA is to the radius, and therefore R×Co-S, B=Co-T, BC×T, BA.

Again, Let the complement of the angle B be the middle part, and the complement of the angle C, and the ſide AC will be oppoſite extremes: and by prop. 22. Co-S, AC is to the radius, as Co-S, B is to S, C: and therefore R×Co-S, B=Co-S, AC×S, C.

Thirdly, Let the complement of the hypothenuſe be the middle part, and the complements of the angles B, C, will be adjacent extremes; but by cor. 2. prop. 19. Co-S, BC is to Co-T, C as Co-T, B to the radius: therefore R×Co-S, BC=Co-T, B×Co-T, C.

Again, Let the complement of the hypothenuſe be the middle part, and the ſides AB, AC will be oppoſite extremes: but by prop. 21. Co-S, AC is to the radius, as Co-S, BC to Co-S, BA; therefore R×Co-S, BC=Co-S, BA×Co-S, AC. Q. E. D.

Fig. 16.

SOLUTION of the Sixteen CASES of Right-Angled Spherical Triangles.

GENERAL PROPOSITION.

IN a right-angled ſpherical triangle, of the three ſides, and three angles, any two being given beſides the right angle, the other three may be found.

In the following table the ſolutions are derived from the preceding propoſitions. It is obvious that the ſame ſolutions may be derived from Baron Napier's two rules above demonſtrated, which, as they are eaſily remembered, are commonly uſed in practice.

Caſe	Given	Sought	
1	AC, C	B	R : CoS, AC : : S, C : CoS, B. And B is of the ſame ſpecies with CA, by 22. and 13.
2	AC, B	C	CoS, AC : R : : CoS, B : S, C. By 22.
3	B, C	AC	S, C : CoS, B : : R : CoS, AC. By 22. and AC is of the ſame ſpecies with B. 13.
4	BA, AC	BC	R : CoS, BA : : CoS, AC : CoS, BC 21. and if both BA, AC be greater or leſs than a quadrant, BC will be leſs than a quadrant. But if they be of different affection, BC will be greater than a quadrant. 14.
5	BA, BC	AC	CoS, BA : R : : CoS, BC : CoS, AC 21. and if BC be greater or leſs than a quadrant, BA, AC will be of different or the ſame affection By 15.
6	BA, AC	B	S, BA : R : : T, CA : T, B, 17. and B is of the ſame affection with AC. 13.
7	BA, B	AC	R : S, BA : : T, B : T, AC. 17. And AC is of the ſame affection with B. 13.

Caſe	Given	Sought	
8	AC, B	BA	T, B : R : : T, CA : S, BA. 17.
9	BC, C	AC	R : CoS, C : : T, BC : T, CA. 20. If BC be leſs or greater than a quadrant, C and B will be of the ſame or different affection. 15. 13.
10	AC, C	BC	CoS, C : R : : T, AC : T, BC. 20. And BC is leſs or greater than a quadrant, according as C and AC or C and B are of the ſame or different affections. 14. 1.
11	BC, CA	C	T, BC : R : : T, CA : CoS, C. 20. If BC be leſs or greater than a quadrant, CA and AB, and therefore CA and C, are of the ſame or different affection. 15.
12	BC, B	AC	R : S, BC : : S, B : S, AC. 18. And AC is of the ſame affection with B.
13	AC, B	BC	S, B : S, AC : : R : S, BC. 18.
14	BC, AC	B	S, BC : R : : S, AC : S, B. 18. And B is of the ſame affection with AC.
15	B, C	BC	T, C : R : : CoT, B : CoS, BC. 19. And according as the angles B and C are of different or the ſame affection, BC will be greater or leſs than a quadrant. 14.
16	BC, C	B	R : CoS, BC : : T, C : CoT, B. 19. If BC be leſs or greater than a quadrant, C and B will be of the ſame or different affection. 15.

The ſecond, eighth, and thirteenth caſes, which are commonly called ambiguous, admit of two ſolutions: for in theſe it is not determined whether the ſide or meaſure of the angle ſought be greater or leſs than a quadrant.

PROP. XXIII. Fig. 16.

IN ſpherical triangles, whether right-angled, or oblique-angled, the ſines of the ſides are proportional to the ſines of the angles oppoſite to them.

Firſt, Let ABC be a right-angled triangle, having a right angle at A; therefore by prop. 18. the ſine of the hypothenuſe BC is to the radius (or the ſine of the right angle at A) as the ſine of the ſide AC to the ſine of the angle B. And, in like manner, the ſine of BC is to the ſine of the angle A, as the ſine of AB to the ſine of the angle C; wherefore (11. 5.) the ſine of the ſide AC is to the ſine of the angle B, as the ſine of AB to the ſine of the angle C.

Secondly, Let BCD be an oblique-angled triangle, the ſine of either of the ſides BC, will be to the ſine of either of the other two CD, as the ſine of the angle D oppoſite to BC is to the ſine of the angle B oppoſite to the ſide CD. Through the point C, let there be drawn an arch of a great circle CA perpendicular upon BD; and in the right-angled triangle ABC (18. of this) the ſine of BC is to the radius, as the ſine of AC to the ſine of the angle B; and in the triangle ADC (by 18. of this) and, by inverſion, the radius is to the ſine of DC as the ſine of the angle D to the ſine of AC: therefore, ex aequo perturbate, the ſine of BC is to the ſine of DC, as the ſine of the angle D to the ſine of the angle B. Q. E. D. Fig. 17. 18.

PROP. XXIV. Fig. 17. 18.

IN oblique-angled ſpherical triangles, having drawn a perpendicular arch from any of the angles upon the oppoſite ſide, the co-ſines of the angles at the baſe are proportional to the ſines of the vertical angles.

Let BCD be a triangle, and the arch CA perpendicular to the baſe BD; the co-ſine of the angle B will be to the co-ſine of the angle D, as the ſine of the angle BCA to the ſine of the angle DCA.

For by 22. the co-ſine of the angle B is to the ſine of the angle BCA as (the co-ſine of the ſide AC is to the radius; that is, by prop. 22. as) the co-ſine of the angle D to the ſine of the angle DCA; and, by permutation, the co-ſine of the angle B is to the co-ſine of the angle D, as the ſine of the angle BCA to the ſine of the angle DCA. Q. E. D.

PROP. XXV. FIG. 17. 18.

THE ſame things remaining, the co-ſines of the ſides BC, CD, are proportional to the co-ſines of the baſes BA, AD.

For by 21. the co-ſine of BC is to the co-ſine of BA, as (the co-ſine of AC to the radius; that is, by 21. as) the co-ſine of CD is to the co-ſine of AD: wherefore, by permutation, the co-ſines of the ſides BC, CD are proportional to the co-ſines of the baſes BA, AD. Q. E. D.

PROP. XXVI. FIG. 17. 18.

THE ſame conſtruction remaining, the ſines of the baſes BA, AD are reciprocally proportional to the tangents of the angles B and D at the baſe.

For by 17. the ſine of BA is to the radius, as the tangent of AC to the tangent of the angle B; and by 17. and inverſion, the radius is to the ſine of AD, as the tangent of D to the tangent of AC: therefore, ex aequo perturbate, the ſine of BA is to the ſine of AD, as the tangent of D to the tangent of B.

PROP. XXVII. FIG. 17. 18.

THE co-ſines of the vertical angles are reciprocally proportional to the tangents of the ſides.

For by prop. 20. the co-ſine of the angle BCA, is to the radius as the tangent of CA is to the tangent of BC; and by the ſame prop. 20. and by inverſion, the radius is to the co-ſine of the angle DCA, as the tangent of DC to the tangent of CA: therefore, ex aequo perturbate, the co-ſine of the angle BCA is to the co-ſine of the angle DCA, as the tangent of DC is to the tangent of BC. Q. E. D.

LEMMA. FIG. 19. 20.

IN right-angled plane triangles, the hypothenuſe is to the radius, as the exceſs of the hypothenuſe above either of the ſides to the verſed ſine of the acute angle adjacent to that ſide, or as the ſum of the hypothenuſe, and either of the ſides to the verſed ſine of the exterior angle of the triangle.

Let the triangle ABC have a right angle at B; AC will be to the radius as the exceſs of AC above AB, to the verſed ſine of the angle A adjacent to AB; or as the ſum of AC, AB to the verſed ſine of the exterior angle CAK.

With any radius DE, let a circle be deſcribed, and from D the center let DF be drawn to the circumference, making the angle EDF equal to the angle BAC, and from the point F, let FG be drawn perpendicular to DE: let AH, AK be made equal to AC, and DL to DE: DG therefore is the co-ſine of the angle EDF or BAC, and GE its verſed ſine: and becauſe of the equiangular triangles ACB, DFG, AC or AH is to DF or DE, as AB to DG: therefore (19. 5.) AC is to the radius DE as BH to GE, the verſed ſine of the angle EDF or BAC: and ſince AH is to DE, as AB to DG, (12. 5.) AH or AC will be to the radius DE as KB to LG, the verſed ſine of the angle LDF or KAC. Q. E. D.

PROP. XXVIII. FIG. 21. 22.

IN any ſpherical triangle, the rectangle contained by the ſines of two ſides, is to the ſquare of the radius, as the exceſs of the verſed ſines of the third ſide or baſe, and the arch, which is the exceſs of the ſides, is to the verſed ſine of the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle, the rectangle contained by the ſines of AB, BC will be to the ſquare of the radius, as the exceſs of the verſed ſines of the baſe AC, and of the arch, which is the exceſs of AB, BC to the verſed ſine of the angle ABC oppoſite to the baſe.

Let D be the center of the ſphere, and let AD, BD, CD be joined, and let the ſines AE, CF, CG of the arches AB, BC, AC be drawn; let the ſide BC be greater than BA, and let BH be made equal to BC: AH will therefore be the exceſs of the ſides BC, BA; let HK be drawn perpendicular to AD, and ſince AG is the verſed ſine of the baſe AC, and AK the verſed ſine of the arch AH, KG is the exceſs of the verſed ſines of the baſe AC, and of the arch AH, which is the exceſs of the ſides BC, BA: let GL likewiſe be drawn parallel to KH, and let it meet FH in L, let CL, DH be joined, and let AD, FH meet each other in M.

Since therefore in the triangles CDF, HDF, DC, DH are equal, DF is common, and the angle FDC equal to the angle FDH, becauſe of the equal arches BC, BH, the baſe HF will be equal to the baſe FC, and the angle HFD equal to the right angle CFD: the ſtraight line DF therefore (4. 11.) is at right angles to the plane CFH: wherefore the plane CFH is at right angles to the plane BDH, which paſſes through DF. (18. 11.) In like manner, ſince DG is at right angles to both GC and GL, DG will be perpendicular to the plane CGL; therefore the plane CGL is at right angles to the plane BDH, which paſſes through DG; and it was ſhewn, that the plane CFH or CFL, was perpendicular to the ſame plane BDH; therefore the common ſection of the planes CFL, CGL, viz. the ſtraight line GL, is perpendicular to the plane BDA, (19. 11.) and therefore CLF is a right angle: in the triangle CFL having the right angle CLF, by the Lemma, CF is to the radius as LH, the exceſs, viz. of CF or FH above FL, is to the verſed ſine of the angle CFL; but the angle CFL is the inclination of the planes BCD, BAD, ſince FC, FL are drawn in them at right angles to the common ſection BF: the ſpherical angle ABC is therefore the ſame with the angle CFL; and therefore, CF is to the radius as LH to the verſed ſine of the ſpherical angle ABC; and ſince the triangle AED is equiangular (to the triangle MFD, and therefore) to the triangle MGL, AE will be to the radius of the ſphere AD, (as MG to ML; that is, becauſe of the parallels, as) GK to LH: the ratio therefore which is compounded of the ratios of AE to the radius, and of CF to the ſame radius; that is, (23. 6.) the ratio of the rectangle contained by AE, CF to the ſquare of the radius, is the ſame with the ratio compounded of the ratio of GK to LH, and the ratio of LH to the verſed ſine of the angle ABC; that is, the ſame with the ratio of GK to the verſed ſine of the angle ABC; therefore, the rectangle contained by AE, CF, the ſines of the ſides AB, BC, is to the ſquare of the radius as GK, the exceſs of the verſed ſines AG, AK, of the baſe AC, and the arch AH, which is the exceſs of the ſides to the verſed ſine of the angle ABC oppoſite to the baſe AC. Q. E. D.

PROP. XXIX. FIG. 23.

THE rectangle contained by half of the radius, and the exceſs of the verſed ſines of two arches, is equal to the rectangle contained by the ſines of half the ſum, and half the difference of the ſame arches.

Let AB, AC be any two arches, and let AD be made equal to AC the leſs; the arch DB therefore is the ſum, and the arch CB the difference of AC, AB; through E the center of the circle, let there be drawn a diameter DEF, and AE joined, and CD likewiſe perpendicular to it in G; and let BH be perpendicular to AE, and AH will be the verſed ſine of the arch AB, and AG the verſed ſine of AC, and HG the exceſs of theſe verſed ſines: let BD, BC, BF be joined, and FC alſo meeting BH in K.

Since therefore BH, CG are parallel, the alternate angles BKC, KCG will be equal; but KCG is in a ſemicircle, and therefore a right angle; therefore BKC is a right angle; and in the triangles DFB, CBK, the angles FDB, BCK in the ſame ſegment are equal, and FBD, BKC are right angles; the triangles DFB, CBK are therefore equiangular; wherefore DF is to DB, as BC to CK, or HG; and therefore the rectangle contained by the diameter DF, and HG is equal to that contained by DB, BC; wherefore the rectangle contained by a fourth part of the diameter, and HG, is equal to that contained by the halves of DB, BC: but half the chord DB is the ſine of half the arch DAB, that is, half the ſum of the arches AB, AC; and half the chord of BC is the ſine of half the arch BC, which is the difference of AB, AC. Whence the propoſition is manifeſt.

PROP. XXX. FIG. 19. 24.

THE rectangle contained by half of the radius, and the verſed ſine of any arch, is equal to the ſquare of the ſine of half the ſame arch.

Let AB be an arch of a circle, C its center, and AC, CB, BA being joined; let AB be biſected in D, and let CD be joined, which will be perpendicular to BA, and biſect it in E. (4. 1.) BE or AE therefore is the ſine of the arch DB or AD, the half of AB: let BF be perpendicular to AC, and AF will be the verſed ſine of the arch BA; but, becauſe of the ſimilar triangles CAE, BAF, CA is to AE as AB, that is, twice AE to AF; and by halving the antecedents, half of the radius CA is to AE the ſine of the arch AD, as the ſame AE to AF the verſed ſine of the arch AB. Wherefore by 16. 6. the propoſition is manifeſt.

PROP. XXXI. FIG. 25.

IN a ſpherical triangle, the rectangle contained by the ſines of the two ſides, is to the ſquare of the radius, as the rectangle contained by the ſine of the arch which is half the ſum of the baſe, and the exceſs of the ſides, and the ſine of the arch, which is half the difference of the ſame to the ſquare of the ſine of half the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle, of which the two ſides are AB, BC, and baſe AC, and let the leſs ſide BA be produced, ſo that BD ſhall be equal to BC: AD therefore is the exceſs of BC, BA; and it is to be ſhewn, that the rectangle contained by the ſines of BC, BA is to the ſquare of the radius, as the rectangle contained by the ſine of half the ſum of AC, AD, and the ſine of half the difference of the ſame AC, AD to the ſquare of the ſine of half the angle ABC oppoſite to the baſe AC.

Since by prop. 28. the rectangle contained by the ſines of the ſides BC, BA is to the ſquare of the radius, as the exceſs of the verſed ſines of the baſe AC and AD, to the verſed ſine of the angle B; that is, (1. 6.) as the rectangle contained by half the radius, and that exceſs, to the rectangle contained by half the radius, and the verſed ſine of B; therefore (29. 30. of this,) the rectangle contained by the ſines of the ſides BC, BA is to the ſquare of the radius, as the rectangle contained by the ſine of the arch, which is half the ſum of AC, AD, and the ſine of the arch which is half the difference of the ſame AC, AD is to the ſquare of the ſine of half the angle ABC. Q. E. D.

SOLUTION OF THE TWELVE CASES OF OBLIQUE-ANGLED SPHERICAL TRIANGLES.

GENERAL PROPOSITION.

IN any oblique-angled ſpherical triangle, of the three ſides and three angles, any three being given, the other three may be found.

Fig. 26. 27.

	Given.	Sought.	
1	B, D, and BC, two angles and a ſide oppoſite one of them.	C.	CoS, BC : R : : Co-T, B : T, BCA, 19. Likewiſe by 24. CoS, B : S, BCA : : CoS, D : S, DCA; wherefore BCD is the ſum or difference of the angles DCA, BCA according as the perpendicular CA falls within or without the triangle BCD; that is, (16. of this,) according as the angles B, D, are of the ſame or different affection.
2	B, C, and BC two angles and the ſide between them.	D.	CoS, BC : R : : Co-T, B : T, BCA. 19. and alſo by 24. S, BCA : S, DCA : : CoS, B : CoS, D; and according as the angle BCA is leſs or greater than BCD, the perpendicular CA falls within or without the triangle BCD; and therefore (16. of this,) the angles B, D will be of the ſame or different affection.
3	BC, CD, and B.	BD.	R : CoS, B : : T, BC : T, BA. 20. and CoS, BC : CoS, BA : : CoS, DC : CoS, DA. 25. and BD is the ſum or difference of BA, DA.
4	BC, DB and B.	CD.	R : CoS, B : : T, BC : T, BA. 20. and CoS, BA : CoS, BC : : CoS, DA : CoS, DC. 25. and according as DA, AC are of the ſame or different affection, DC will be leſs or greater than a quadrant. 14.

	Given.	Sought.	
5	B, D and BC.	DB.	R : CoS, B : : T, BC : T, BA. 20. and T, D : T, B : : S, BA : S, DA. 26. and BD is the ſum or difference of BA, DA.
6	BC, BD and B.	D.	R : CoS, B : : T, BC : T, BA. 20. and S, DA : S, BA : : T, B : T, D; and according as BD is greater or leſs than BA, the angles B, D are of the ſame or different affection. 16.
7	BC, DC and B.	C.	CoS, BC : R : : Co-T, B : T, BCA. 19. and T, DC : T, BC : : CoS, BCA : CoS, DCA. 27. the ſum or difference of the angles BCA, DCA is equal to the angle BCD.
8	B, C and BC.	DC.	CoS, BC : R : : Co-T, B : T, BCA. 19. alſo by 27. CoS, DCA : CoS, BCA : : T, BC : T, DC. 27. if DCA and B be of the ſame affection; that is, (13.) if AD and CA be ſimilar, DC will be leſs than a quadrant, 14. and if AD, CA be not of the ſame affection, DC is greater than a quadrant. 14.
9	BC, DC and B.	D.	S, CD : S, B : : S, BC : S, D.
10	B, D and BC.	DC.	S, D : S, BC : : S, B : S, DC.
11	BC, BA, AC. Fig. 25.	B.	S, AB×S, BC : Rq : : S, $\frac{AC+AD}{2}$ ×S, $\frac{AC-AD}{2}$: Sq, $\frac{ABC}{2}$. See Fig. 25. AD being the difference of the ſides BC, BA.

	Given.	Sought.	
12	A, B, C. Fig. 7.	The ſides.	See Fig. 7. In the triangle DEF, DE, EF, FD are reſpectively the ſupplements of the meaſures of the given angles B, A, C in the triangle BAC; the ſides of the triangle DEF are therefore given, and by the preceding caſe the angles D, E, F may be found, and the ſides BC, BA, AC are the ſupplements of the meaſures of theſe angles.

The 3d, 5th, 7th, 9th, 10th, caſes, which are commonly called ambiguous, admit of two ſolutions, either of which will anſwer the conditions required; for, in theſe caſes, the meaſure of the angle or ſide ſought, may be either greater or leſs than a quadrant, and the two ſolutions will be ſupplements to each other. (cor. to def. 4. 6. Pl. Tr.)

If from any of the angles of an oblique-angled ſpherical triangle, a perpendicular arch be drawn upon the oppoſite ſide, moſt of the caſes of oblique-angled triangles may be reſolved by means of Napier's Rules.

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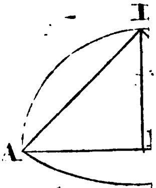

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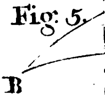

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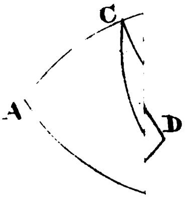

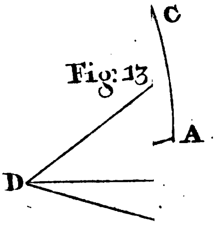

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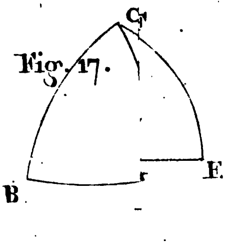

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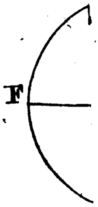

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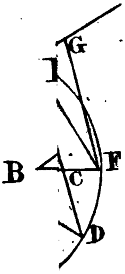

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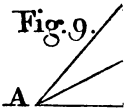

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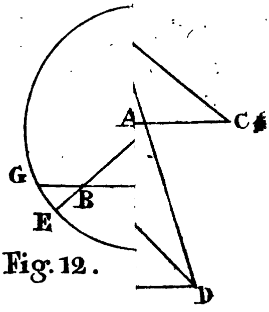
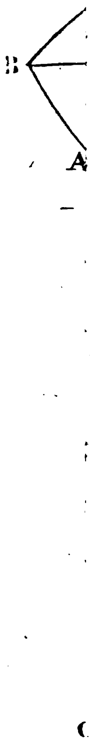


Fig. 12.

16.


B
A
C